Groups and Group Actions, Sheet 1, HT20

Binary Operations. The Group Axioms. Examples.

Main course

1. For each of the following sets S and binary operations $*$ on S, state whether (a) $*$ is associative, (b) $*$ is commutative, (c) an identity exists, (d) inverses exist.

(i) $S = \mathbb{N}$, $m * n = \max\{m, n\}$.
(ii) $S = \mathbb{Z}$, $m * n = m + n + 1$.
(iii) $S = M_n(\mathbb{R})$ where $n \geqslant 2$ and $A * B = \frac{1}{2}(AB + BA)$.
(iv) $S = \{f : \mathbb{R} \to \mathbb{R}\}$, $f * g = f \circ g$.

2. There are 56 different Latin squares of size 5×5 whose first row and first column are both (e, a, b, c, d). Construct one such square which is a group table, justifying your reasons for it representing a group, and one such square that does not represent a group, again saying why.

3. Let A and B be complex $n \times n$ matrices. If $A = (a_{ij})$ then we define its complex conjugate as $\overline{A} = (\overline{a_{ij}})$. Show that

$$\overline{A + B} = \overline{A} + \overline{B}, \qquad \overline{AB} = \overline{A}\,\overline{B}.$$

Show that $U(n)$ is a group. Show further that $U(1)$ is Abelian and that $U(n)$ is non-Abelian for $n \geqslant 2$.

4. An *affine transformation* of $\mathbb{R}^2$ is one of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b}$$

where A is an invertible 2×2 matrix and $\mathbf{b}$ is a 2×1 column vector. Let g_1 and g_2 be affine transformations of $\mathbb{R}^2$. Show that their composition $g_2 \circ g_1$ is an affine transformation. Show further that the affine transformations of $\mathbb{R}^2$ form a group $AGL(2, \mathbb{R})$ under composition.

5. The following Cayley table describes a group G. (You are not asked to prove this.)

$*$	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	e	h	g	f	d	c	b
b	b	c	d	f	g	h	e	a
c	c	b	a	e	h	g	f	d
d	d	f	g	h	e	a	b	c
f	f	d	c	b	a	e	h	g
g	g	h	e	a	b	c	d	f
h	h	g	f	d	c	b	a	e

(i) Find the inverse of each element of G.
(ii) Are there any elements of G, other than e, which commute with every element of G?
(iii) Determine the order of each element of G.
(iv) Show that $G = \{e, b, b^2, b^3, a, ba, b^2a, b^3a\}$ and that $ab = b^3a$.
(v) In all, G has ten subgroups, of orders $1, 2, 2, 2, 2, 2, 4, 4, 4, 8$. List the subgroups of G.

6. (i) Let G, H be groups. Show that $G \times H$ is Abelian if and only if G, H are both Abelian.
(ii) Show that the map

$$\phi : \mathbb{C}^* \to (0, \infty) \times S^1 \qquad \text{given by} \qquad \phi(z) = (|z|, z/|z|)$$

is an isomorphism.
(iii) Show that S^1 is isomorphic to $SO(2)$ but that $S^1 \times \{\pm 1\}$ is not isomorphic to $O(2)$.

Starter

S1. Show that the general linear group $GL_n(\mathbb{R})$ (of invertible $n \times n$ real matrices) does indeed form a group under matrix multiplication.

S2. Using the notation from the lecture notes, show that e, r, r^2, r^3, s, rs, r^2s, r^3s are 8 distinct symmetries of the square.

S3. Complete the proof of Proposition 5 from the notes by showing that the product operation $*$ on $G \times H$ is associative.

Pudding

P1. Let G be a finite group of even order. Must it contain an element of order 2?

P2.

(i) Is there a group in which no non-identity element is its own inverse?

(ii) Is there a group in which every non-identity element is its own inverse?

P3. How many ways are there to complete the following grid so that it is the Cayley table of a group?

$*$	e	a	b	c
e				
a				
b				
c				