Groups and Group Action, Sheet 1, HT2020

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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> **S1.** Show that the general linear group $GL_n(\mathbb{R})$ (of invertible $n \times n$ real matrices) does indeed form a group under matrix multiplication.

- closed under matrix multiplication: if A and B are invertible, then we know (from Linear Algebra I) that AB is invertible.
- identity: $I \in GL_n(\mathbb{R})$.
- associativity: matrix multiplication is associative.
- inverses: if $A \in GL_n(\mathbb{R})$ then A is invertible. Then $AA^{-1} = I = A^{-1}A$, so also A^{-1} is invertible and hence $A^{-1} \in GL_n(\mathbb{R})$.

> **S2.** Using the notation from the lecture notes, show that e, r, r^2, r^3, s, rs, r^2s, r^3s are 8 distinct symmetries of the square.

As with the example for the equilateral triangle in the notes, for each symmetry we see how it permutes the vertices.

$$e : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$
$$r : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$
$$r^2 : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$
$$r^3 : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$
$$s : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$
$$rs : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$
$$r^2s : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$
$$r^3s : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

These are 8 distinct permutations, so the 8 symmetries are distinct.

S3. Complete the proof of Proposition 5 from the notes by showing that the product operation $*$ on $G \times H$ is associative.

Take (g_1, h_1), (g_2, h_2), $(g_3, h_3) \in G \times H$. Then

$$\begin{aligned}
\big((g_1, h_1) * (g_2, h_2)\big) * (g_3, h_3) &= (g_1 *_G g_2, h_1 *_H h_2) * (g_3, h_3) \\
&= \big((g_1 *_G g_2) *_G g_3, (h_1 *_H h_2) *_H h_3)\big) \\
&= \big(g_1 *_G (g_2 *_G g_3), h_1 *_H (h_2 *_H h_3)\big) \text{ as } *_G \text{ and } *_H \text{ associative} \\
&= (g_1, h_1) * (g_2 *_G g_3, h_2 *_H h_3) \\
&= (g_1, h_1) * \big((g_2, h_2) * (g_3, h_3)\big)
\end{aligned}$$

so $*$ is associative.