Groups and Group Actions, Sheet 2, HT20

Permutations of a finite set. Transpositions. Parity. Conjugacy.

Main course

1. Three permutations α, β, γ of $\{1, 2, \ldots, 11, 12\}$ are given below. In each case express the permutation as a product of disjoint cycles.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 6 & 5 & 7 & 4 & 8 & 11 & 2 & 3 & 9 & 10 & 1 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$
$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 5 & 6 & 1 & 9 & 10 & 8 & 12 & 7 & 11 \end{pmatrix}$$

What is the order of each permutation? Which are even and which are odd permutations? Write α^2, $\alpha\beta$, γ^{-1} as products of disjoint cycles.

2. There are five different types of cycle decomposition of $\{1, 2, 3, 4\}$: the identity permutation, 2-cycles (e.g. $(1\,2)$), 3-cycles (e.g. $(1\,2\,3)$), 4-cycles (e.g. $(1\,2\,3\,4)$) and double transpositions (e.g. $(1\,2)(3\,4)$). How many are there of each type in S_4? (You should have $4! = 24$ permutations in all.)

What types of cycle decomposition arise among even permutations of $\{1, 2, 3, 4, 5\}$ and how many are there of each type? (Note that $|A_5| = 60$.)

3. Let σ be a permutation of $\{1, \ldots, n\}$ and let $k \in \{1, \ldots, n\}$. Show that

$$\sigma^{-1}(1\,2\,3\,\ldots\,k)\sigma = (1\sigma\,2\sigma\,\ldots\,k\sigma).$$

The permutations α, β, $\gamma \in S_5$ are given by

$$\alpha = (1\,2\,3)(4\,5), \quad \beta = (1\,2\,3\,4), \quad \gamma = (2\,3).$$

Find $(\alpha\beta^5\gamma^3\alpha^2\gamma\beta^3\alpha^5)^3$.

4. (i) Let $\mathrm{sgn}(\sigma)$ denote the sign of a permutation $\sigma \in S_n$ (that is, $\mathrm{sgn}(\sigma) = 1$ if σ is even and -1 if σ is odd). Show for σ, $\tau \in S_n$ that

$$\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau).$$

(ii) Show that a permutation with odd order must be even. Does the converse hold?
(iii) The vertices of a regular pentagon are labelled clockwise 1 to 5. Show that every symmetry of the pentagon corresponds to an even permutation. How will the situation vary if the vertices are now arbitrarily labelled with the numbers 1 to 5?

5. (i) Show that $V_4 = \{e, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\}$ is an Abelian group.
(ii) Show that V_4 is isomorphic to $C_2 \times C_2$.
(iii) How many isomorphisms are there from V_4 to $C_2 \times C_2$?

6. (i) Find a permutation $\alpha \in S_7$ such that $\alpha^4 = (2\,1\,4\,3\,5\,6\,7)$. Is α unique?
(ii) Find all permutations $\alpha \in S_7$ such that $\alpha^3 = (1\,2\,3\,4)$.
(iii) Find permutations α, $\beta \in S_5$ both with order 3 such that $\alpha\beta$ has order 5.
(iv) Let $n \geqslant 3$. Find permutations α, $\beta \in S_n$ both with order 2 such that $\alpha\beta$ has order n.

Starter

S1. Here are two permutations in S_{10}. Write each as a product of disjoint cycles.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 6 & 10 & 2 & 7 & 8 & 9 & 4 & 5 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 8 & 5 & 3 & 2 & 9 & 6 & 1 & 7 & 10 \end{pmatrix}$$

What is the order of each?
Find each of the following products (using disjoint cycle notation): $\alpha\beta$, $\beta\alpha$, $\alpha^2\beta$, $\alpha^{-7}\beta^{-5}$, $\alpha\beta^3$.

S2. Which permutations in S_4 are even?

S3. How many permutations in S_6 have order 6?

Pudding

P1. What is the largest possible order of an element in S_5? In S_9? What is the smallest n for which an element of largest order in S_n must have three cycles (in disjoint cycle notation)?

P2. For each of the following sets X_i, determine whether every permutation in S_n can be written as a product of elements of X_i.

(i) $X_1 = \{(j\ j+1) : 1 \leqslant j < n\}$

(ii) $X_2 = \{(1\ k) : 1 < k \leqslant n\}$

(iii) $X_3 = \{(1\ 2), (1\ 2\ \ldots\ n)\}$.

P3. If we choose a permutation in S_n at random, with all permutations being equally likely, what is the probability that our chosen permutation has exactly one 1-cycle?