Groups and Group Action, Sheet 2, HT2020

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. Here are two permutations in S_{10}. Write each as a product of disjoint cycles.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 6 & 10 & 2 & 7 & 8 & 9 & 4 & 5 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 8 & 5 & 3 & 2 & 9 & 6 & 1 & 7 & 10 \end{pmatrix}$$

What is the order of each?
Find each of the following products (using disjoint cycle notation): $\alpha\beta$, $\beta\alpha$, $\alpha^2\beta$, $\alpha^{-7}\beta^{-5}$, $\alpha\beta^3$.

We have $\alpha = (2\ 3\ 6\ 7\ 8\ 9\ 4\ 10\ 5)$ and $\beta = (1\ 4\ 3\ 5\ 2\ 8)(6\ 9\ 7)$.
This shows that α has order 9, and β has order 6 (the least common multiple of 6, 3 and 1).
We have

$$\begin{aligned}
\alpha\beta &= (1\ 4\ 10\ 2\ 5\ 8\ 7)(3\ 9) \\
\beta\alpha &= (1\ 10\ 5\ 3\ 2\ 9\ 8)(4\ 6) \\
\alpha^2\beta &= \alpha(\alpha\beta) = (1\ 4\ 2\ 9\ 10\ 8\ 3\ 6) \\
\alpha^{-7}\beta^{-5} &= \alpha^2\beta \text{ since } \alpha^9 = e = \beta^6 \\
\alpha\beta^3 &= (2\ 3\ 6\ 7\ 8\ 9\ 4\ 10\ 5)(1\ 4\ 3\ 5\ 2\ 8)^3 \text{ since disjoint cycles commute and } (6\ 9\ 7)^3 = e \\
&= (1\ 5\ 4\ 10)(2\ 8\ 9)(3\ 6\ 7).
\end{aligned}$$

[You want to develop your fluency so that you can find these products using disjoint cycle notation, rather than going back to the rather clumsy notation used to define α and β in this question.]

S2. Which permutations in S_4 are even?

The possible cycle types in S_4 are listed in Q2 of the main course of this sheet. A k-cycle is even if and only if k is odd, and we need an even number of odd cycles in the cycle type. ("A permutation is even if and only if its cycle type has an even number of cycles of even length.")

So we see that the cycle types in S_4 corresponding to even permutations are the identity, the 3-cycles, and the double transpositions.

S3. How many permutations in S_6 have order 6?

A permutation in S_6 has order 6 if it has cycle type 6 (a single 6-cycle) or 3, 2, 1.

There are $5! = 120$ 6-cycles in S_6: we may assume that 1 appears first, since cycling the elements within a cycle doesn't change it, and then there are 5! ways to arrange the remaining elements.

Now we count the elements with cycle type 3, 2, 1. There are 6 possible elements to go in the 1-cycle. Then we can choose the two elements for the transposition in $\binom{5}{2} = 10$ ways. Finally, there are 2 possible 3-cycles made up of the remaining elements (for example $(1\ 2\ 3)$ and $(1\ 3\ 2)$). So there are $6 \times 10 \times 2 = 120$ elements of this cycle type.

This gives a total of $120 + 120 = 240$ elements in S_6 with order 6.