Groups and Group Actions, Sheet 3, HT20

Subgroups. Cyclic Groups. hcf and lcm. Equivalence relations.

Main course

1. For each part below, provide a group G and a proper, non-trivial subgroup H of G according to the different criteria. Provide a different group G in each case.
(i) G is infinite and H is infinite and cyclic;
(ii) G is infinite and H is finite and cyclic;
(iii) G is infinite and non-Abelian and H is Abelian;
(iv) G is infinite and Abelian and H is infinite and not cyclic;
(v) G is infinite and non-Abelian and H is infinite and cyclic.

2. Let G be a group and H, K subgroups of G. Let $HK = \{hk : h \in H, k \in K\}$.
(i) Show that $H \cap K$ is a subgroup of G.
(ii) Give an example where $H \cup K$ is not a subgroup of G. Justify your answer.
(iii) Give an example where HK is not a subgroup of G. Justify your answer.
(iv) Show that if G is Abelian then HK is a subgroup of G .

3. Which of the following groups are cyclic? Either find a generator or show that no generator exists. For the cyclic groups, determine how many different generators there are.

$$\mathbb{Z}^2; \qquad C_2 \times C_4; \qquad C_3 \times C_4; \qquad \langle (1\,2)(3\,4)(5\,6), (1\,4\,5)(2\,3\,6) \rangle \leqslant S_6; \qquad \langle (1\,2\,3), (4\,5\,6) \rangle \leqslant S_6.$$

4. (i) By considering partitions, calculate the number of equivalence relations on a set with four elements.

(ii) Let $q_0 = 1$ and, for $n \geqslant 1$, let q_n denote the number of equivalence relations of a set X with n elements. By considering the possible equivalence classes of the $(n+1)$th element, show that

$$q_{n+1} = \sum_{k=0}^{n} \binom{n}{k} q_{n-k}.$$

Use this recurrence relation to verify your answer for q_4 from part (i).

5. Let G be a finite group. We define a relation $\sim$ on $G \backslash \{e\} = \{g \in G : g \neq e\}$ by

$$g \sim h \text{ if and only there exists } k \text{ such that } g = h^k.$$

(i) Show that $\sim$ is reflexive and transitive.
(ii) Show that $\sim$ is symmetric if and only if the order of g is prime for every $g \neq e$.

6 Let G be a finite group and H, K subgroups of G. The map $\phi : H \times K \to HK$ is defined by $(h,k) \to hk$.
(i) Show that $|\phi^{-1}(g)| = |H \cap K|$ for any $g \in HK$. Deduce that

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}.$$

(ii) Show that if G is Abelian and $H \cap K = \{e\}$ then HK is isomorphic to $H \times K$.

Starter

S1. In the dihedral group D_8 (with the same notation as in lectures), what is the subgroup generated by rs? What is the subgroup $\langle rs, s\rangle$?

S2. In this question we work in the group $\mathbb{Z}$. Find all integers m such that $\langle m\rangle = \langle 3, 4, 6\rangle$, and find all integers n such that $\langle n\rangle = \langle 3\rangle \cap \langle 4\rangle \cap \langle 6\rangle$.

S3. Let G be a group. Define a relation $\sim$ on G via $x \sim y$ if and only if $x = y$ or $x = y^{-1}$. Show that $\sim$ is an equivalence relation. What are the equivalence classes?

Pudding

P1. Let G be a group with at least 2 elements. Suppose that G has no proper non-trivial subgroups. Take $x \in G \setminus \{e\}$. Is x a generator for G? Is x^2 a generator for G? What can you say about the order of x?

P2.

(i) Is there a non-cyclic group all of whose proper subgroups are cyclic?

(ii) Is there a non-Abelian group all of whose proper subgroups are Abelian?

P3. Let m, $n \in \mathbb{Z}$. What can you say about $\mathrm{hcf}(m, n) \times \mathrm{lcm}(m, n)$ (the product of their highest common factor and least common multiple)?