Groups and Group Action, Sheet 3, HT2020

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. In the dihedral group D_8 (with the same notation as in lectures), what is the subgroup generated by rs? What is the subgroup $\langle rs, s\rangle$?

We have $\langle rs\rangle = \{(rs)^k : k \in \mathbb{Z}\}$.

But $(rs)^2 = e$, so we find that $\langle rs\rangle = \{e, rs\} \leqslant D_8$.

Now $\langle rs, s\rangle$ is the smallest subgroup containing rs and s, so it contains all products involving rs and s. In particular, it contains their product $rss = r$. So $\langle rs, s\rangle$ contains both r and s. But these generate all elements of the group, so $\langle rs, s\rangle = D_8$.

S2. In this question we work in the group $\mathbb{Z}$. Find all integers m such that $\langle m\rangle = \langle 3, 4, 6\rangle$, and find all integers n such that $\langle n\rangle = \langle 3\rangle \cap \langle 4\rangle \cap \langle 6\rangle$.

Let m be an integer such that $\langle m\rangle = \langle 3, 4, 6\rangle$. Then $1 = 4 - 3 \in \langle m\rangle$, and since 1 generates $\mathbb{Z}$ this means that $\langle m\rangle = \mathbb{Z}$. We see that the only possibilities for m are 1 and -1.

Let n be an integer such that $\langle n\rangle = \langle 3\rangle \cap \langle 4\rangle \cap \langle 6\rangle$. Now $\langle 6\rangle \subseteq \langle 3\rangle$ (every multiple of 6 is a multiple of 3), so $\langle n\rangle = \langle 4\rangle \cap \langle 6\rangle = \langle 12\rangle$. We see that $n = 12$ or $n = -12$.

S3. Let G be a group. Define a relation $\sim$ on G via $x \sim y$ if and only if $x = y$ or $x = y^{-1}$. Show that $\sim$ is an equivalence relation. What are the equivalence classes?

- reflexive: for $x \in G$ we have $x = x$ and so $x \sim x$.
- symmetric: if $x \sim y$ then $x = y$ or $x = y^{-1}$, so $y = x$ or $y = x^{-1}$, so $y \sim x$.
- transitive: if $x \sim y$ and $y \sim z$, then we have four cases to check. If $x = y$ and $y = z$, then $x = z$ so $x \sim z$. If $x = y$ and $y = z^{-1}$, then $x = z^{-1}$ so $x \sim z$. If $x = y^{-1}$ and $y = z$, then $x = z^{-1}$ so $x \sim z$. Finally, if $x = y^{-1}$ and $y = z^{-1}$ then $x = z$ so $x \sim z$.

What is the equivalence class of a fixed element x? If $x \neq x^{-1}$, then the equivalence class is $[x] = \{x, x^{-1}\}$. If x is self-inverse (that is, has order 1 or 2), then $[x] = \{x\}$.