Groups and Group Actions, Sheet 3, HT2020
Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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> **P1.** Let G be a group with at least 2 elements. Suppose that G has no proper non-trivial subgroups. Take $x \in G \setminus \{e\}$. Is x a generator for G? Is x^2 a generator for G? What can you say about the order of x?

Let's consider $H = \langle x \rangle$. This is a subgroup of G. But G has no proper non-trivial subgroups, and $x \neq e$ so $H \neq \{e\}$, so $H = G$. That is, $G = \langle x \rangle$, so x generates G.

If $x^2 \neq e$, then exactly the same argument (applied to x^2 in place of x) shows that x^2 generates G. If $x^2 = e$, then $\langle x^2 \rangle = \{e\} \subsetneq G$. But also we know that x generates G, so then $G = \{e, x\}$.

I claim that if x has finite order then the order of x is a prime.

Let d be the order of x, so also $|G| = d$ (as x generates G). If $d = ab$ with $b > 1$, then consider $\langle x^a \rangle$. Since $a < d$ we know that $x^a \neq e$, so $\langle x^a \rangle$ is a non-trivial subgroup of G so must be G. But x^a has order b, so we need $b = d$ and hence $a = 1$. So d is prime.

If x does not have finite order, then G is infinite and cyclic, so is isomorphic to $\mathbb{Z}$. But $\mathbb{Z}$ does have proper non-trivial subgroups. So this case does not arise.

> **P2.**
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> (i) Is there a non-cyclic group all of whose proper subgroups are cyclic?
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> (ii) Is there a non-Abelian group all of whose proper subgroups are Abelian?

(i) Consider $C_2 \times C_2$. This is not cyclic (it has order 4, but every element has order at most 2). But its proper subgroups are all cyclic.

(ii) Consider S_3, which is not Abelian. We see that its proper subgroups are all Abelian.

P3. Let m, $n \in \mathbb{Z}$. What can you say about $\mathrm{hcf}(m,n) \times \mathrm{lcm}(m,n)$ (the product of their highest common factor and least common multiple)?

Perhaps inspired by some experimentation with numbers, we might conjecture that $\mathrm{hcf}(m,n) \times \mathrm{lcm}(m,n) = mn$. There are many ways to prove this. Here is one strategy, using properties from Proposition 22 in the notes.

Let $h = \mathrm{hcf}(m,n)$ and $l = \mathrm{lcm}(m,n)$. So we are trying to prove that $hl = mn$.

Let $k = \frac{mn}{h}$, so we want to prove that $k = l$.

First we show that k is a common multiple of m and n. Indeed, since $h \mid m$ we see that k is an integer, and moreover $n \mid k$. Similarly, since $h \mid n$ we see that $m \mid k$. So k is a common multiple of m and n, so $l \mid k$.

Now take an arbitrary common multiple of m and n, say c, so $m \mid c$ and $n \mid c$. We want to show that $k \mid c$. Since $h = \mathrm{hcf}(m,n)$, we know that there are integers a, b such that $h = am + bn$. Then

$$\frac{c}{k} = \frac{ch}{mn} = \frac{c(am+bn)}{mn} = \frac{ac}{n} + \frac{bc}{m},$$

and since $m \mid c$ and $n \mid c$, this is an integer. That is, $k \mid c$.

So $k = l$, so indeed $hk = mn$.