Linear Algebra II Problem Sheet 1, HT 2020

1. Calculate the determinants of the five matrices

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 1 & 5 \\ 4 & 2 & 3 \end{pmatrix}, \; AB^2, \; A+B, \; AB+A^2.$$

2. Solve the equation

$$\det \begin{pmatrix} a-x & b-x & c \\ a-x & c & b-x \\ a & b-x & c-x \end{pmatrix} = 0$$

for the variable x.

3. Let A be an $n \times n$ matrix. Suppose that A has the form $\begin{pmatrix} U & V \\ W & X \end{pmatrix}$ in which U, V, W and X are $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$ and $n_2 \times n_2$ matrices respectively, with $n_1 + n_2 = n$. Show that if $W = 0$ then $\det(A) = \det(U)\det(X)$.

4. Show that

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \alpha+\beta & \alpha+\gamma \\ 1 & \beta+\alpha & 0 & \beta+\gamma \\ 1 & \gamma+\alpha & \gamma+\beta & 0 \end{pmatrix} = -4(\alpha\beta + \beta\gamma + \gamma\alpha).$$

What is the value of this when α, β, γ are the three roots of the equation $x^3 - 1 = 0$?

5. Here we derive an explicit formula for the inverse of a matrix with non-zero determinant.

(i) (Cramer's rule) Let $\mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbb{R}^n$ (column vectors) and $x_1, \cdots, x_n \in \mathbb{R}$ with

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

for some $\mathbf{b} \in \mathbb{R}^n$. Show that for each i we have

$$x_i \det[\mathbf{a}_1, \cdots, \mathbf{a}_n] = \det[\mathbf{a}_1, \cdots, \mathbf{b}, \cdots, \mathbf{a}_n]$$

where the $\mathbf{b}$ occurs in the ith place. [Hint: Use the properties of det from Definition 1.1 and Proposition 1.2 of the lecture notes.]

(ii) Now let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ and assume $\det(A) \neq 0$, and so there exists $B = (b_{ij}) \in M_n(\mathbb{R})$ such that $AB = I_n$. Write $\mathbf{e}_j \in \mathbb{R}^n$ for the column vector with 1 in the jth place and zeros elsewhere. Show using (i) that

$$b_{ij} = \frac{\det[\mathbf{a}_1, \cdots, \mathbf{e}_j, \cdots, \mathbf{a}_n]}{\det(A)}$$

where the $\mathbf{e}_j$ occurs in the ith position.

(iii) By expanding $\det[\mathbf{a}_1, \cdots, \mathbf{e}_j, \cdots, \mathbf{a}_n]$ down the ith column using the Laplace expansion, show that

$$b_{ij} = \frac{(-1)^{i+j}\det(A_{ji})}{\det(A)}$$

where A_{ji} is the matrix obtained from A by deleting the jth row and ith column.