**Linear Algebra II Problem Sheet 2, HT 2020**

1. Using elementary row operations, compute

$$\det\begin{pmatrix} 1 & 2 & 3 & 0 \\ 5 & 0 & 2 & 1 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 3 & -2 \end{pmatrix}.$$

2. If $x_1, x_2, \cdots, x_n \in \mathbb{R}$ show by induction that for $n \geq 2$ we have

$$V_n = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

   [Hint: if $c_i$ denotes the $i$th column of $V_n$, then carry out successively the column operations $c_n \mapsto c_n - x_1 c_{n-1}$, $c_{n-1} \mapsto c_{n-1} - x_1 c_{n-2}, \cdots, c_2 \mapsto c_2 - x_1 c_1$, to find that

$$V_n = (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) V'_{n-1}$$

   where $V'_{n-1}$ is the same as $V_{n-1}$ but with $x_1, x_2, \cdots, x_n$ replaced with $x_2, x_3, \cdots, x_n$.]

3. Let $B = (b_{ij})$ be an upper triangular $n \times n$ matrix, so $b_{ij} = 0$ if $i > j$.

   (i) Show that $\det B = \prod_{i=1}^{n} b_{ii}$.

   (ii) Show that $\lambda$ is an eigenvalue of $B$ if and only if it equals $b_{ii}$ for some $i$.

4. For $n \geq 2$ let $J$ be the $n \times n$ matrix all of whose entries are 1.

   (i) Show that $(1, 1, \cdots, 1)^T$ is an eigenvector with eigenvalue $n$.

   (ii) Given that 0 is an eigenvalue, find the eigenvectors with eigenvalue 0.

5. Let $V$ be a finite dimensional real vector space, and $S : V \to V$ a linear mapping with $S^2 = I$. Show that

   (i) if $\lambda$ is an eigenvalue of $S$, then $\lambda = \pm 1$.

   (ii) $V = U \oplus W$, where $U = \{u \in V : Su = u\}$ and $W = \{w \in V : Sw = -w\}$. [*Hint:* $v = \frac{1}{2}(v + Sv) + \frac{1}{2}(v - Sv)$.]

   Deduce that $V$ has a basis with respect to which the matrix of $S$ is the diagonal matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}.$$

   Now suppose that $T : V \to V$ is linear and satisfies $ST = TS$ and $T^2 = I$. Show that $T(U) \subseteq U$ and that $U = X \oplus Y$, where $X = \{u \in U : Tu = u\}$ and $Y = \{u \in U : Tu = -u\}$. Deduce that there exists a basis of $V$ relative to which all three maps $S, T$ and $ST$ are represented by diagonal matrices.

6. Let $E$ be a square matrix over $\mathbb{C}$ such that $E^{k+1} = 0$ for some $k \geq 1$. Show, by explicitly computing an inverse, that the matrix $I - \lambda E$ is invertible for all $\lambda \in \mathbb{C}$. What can you deduce about the eigenvalues of $E$?