Linear Algebra II Problem Sheet 4, HT 2020

1. Let $V = \mathbb{R}^3$ and

$$u_1 = (1, 0, 1),\ u_2 = (2, 3, 2),\ u_3 = (-1, 4, 7).$$

Compute a basis v_1, v_2, v_3 for $\mathbb{R}^3$ which is orthonormal with respect to the dot product such that $\mathrm{Sp}\{u_1, \cdots, u_i\} = \mathrm{Sp}\{v_1, \cdots, v_i\}$ for each $1 \le i \le 3$.

2. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Show that if λ, $\mu \in \mathbb{R}$ are distinct eigenvalues of A with v and w associated eigenvectors, then v and w are orthogonal; that is, $v^T w = 0$.

3. Find a real orthogonal matrix P such that $P^T AP$ is diagonal when A is each of the following matrices

$$\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix},\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

4. Verify that if P is an orthogonal matrix and $x = Py$ then $y^T y = x^T x$.

Let A be a real symmetric $n \times n$ matrix. Then we know that there exists a real orthogonal matrix P such that $P^T AP$ is diagonal. By using the transformation $x = Py$, or otherwise, prove that for every $x \in \mathbb{R}^n$

$$mx^T x \le x^T Ax \le Mx^T x,$$

where m and M are the smallest and greatest eigenvalues of A respectively. For which x is it true that $x^T Ax = Mx^T x$?

Let $A = \begin{pmatrix} 5 & 1 & \sqrt{2} \\ 1 & 5 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 6 \end{pmatrix}$. Find the maximum and minimum values of $x^T x$ for those x for which $x^T Ax = 1$. Giving no heed to orientation, sketch the surface S with equation $x^T Ax = 1$, and indicate on it those vectors x at which $x^T x$ attains its maximum and minimum values on S.

5. Show that for any real $n \times n$ matrix A, $A^T A$ is symmetric.

Suppose that

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

By considering $A^T A$, or otherwise, calculate the maximum and minimum value of $||Ax||$ on the sphere $\{x \in \mathbb{R}^3 : ||x|| = 1\}$.