1. Let $\phi, \psi : [-6,6] \to \mathbb{R}$ be defined by

$$\phi = 3\mathbf{1}_{[-2,2]} - \mathbf{1}_{(0,4)} + \mathbf{1}_{[4,5)},\ \psi = \mathbf{1}_{(0,1]} + \mathbf{1}_{[1,5)}.$$

Here, $\mathbf{1}_X$ means the function taking the value 1 on X and 0 elsewhere.

(i) Sketch the graphs of ϕ and ψ;

(ii) Write down a partition $\mathcal{P}$ to which ϕ is adapted, and hence express ϕ as a linear combination of indicator functions of disjoint bounded intervals;

(iii) Evaluate $I(\phi)$;

(iv) Find a partition $\mathcal{P}$ to which both ϕ and ψ are adapted. Express the step functions

$$|\phi|,\ \phi^3,\ \phi - \psi,\ \max(\phi, \psi)$$

as linear combinations of indicator functions of bounded intervals.

2. Suppose that ϕ is a step function. Show that $|\phi|$ is also a step function and that

$$|I(\phi)| \leqslant I(|\phi|).$$

If $|\phi|$ is a step function, is ϕ a step function? Justify your answer.

3. Fix real numbers a, b with $a < b$. Denote by $\mathscr{L}_{\text{step}}$ be the set of all functions which are step functions on $[a, b]$. Explain why this is a vector space. Which of the following statements are true?

(i) $\mathscr{L}_{\text{step}}$ is the linear span of all indicator functions $\mathbf{1}_{(c,d)}$ of bounded open intervals, restricted to $[a, b]$ (here and below we do *not* assume that $a \leqslant c \leqslant d \leqslant b$);

(ii) $\mathscr{L}_{\text{step}}$ is the linear span of all indicator functions $\mathbf{1}_{[c,d]}$ of bounded closed intervals, restricted to $[a, b]$;

(iii) $\mathscr{L}_{\text{step}}$ is the linear span of all indicator functions $\mathbf{1}_{(c,d]}$ of bounded left-half-open intervals, restricted to $[a, b]$;

(iv) $\mathscr{L}_{\text{step}}$ is finite-dimensional.

4. Let $a < c < d < b$, and $f : [a, b] \to \mathbb{R}$. Prove that if f is Riemann integrable on $[a, b]$, then it is also Riemann integrable on $[c, d]$.

5. Let $S = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$. Define a function $f : [0, 1] \to \mathbb{R}$ by $f(x) = 1$ if $x \in S$ and $f(x) = 0$ otherwise. Show that f is Riemann integrable and that its integral is zero.

6. *Suppose that $f : [0, 1] \to \mathbb{R}$ is an integrable function such that $f(x) > 0$ for all $x \in [0, 1]$. Is it true that $\int_0^1 f > 0$?

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