1. Evaluate $\int_2^5 \frac{dx}{\sqrt{x-1}}$, explaining carefully which results from the course you are using.

2. Define functions $c, s : \mathbb{R} \to \mathbb{R}$ by

$$c(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and

$$s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(i) Show that the series for c, s converge for all x and that both functions are differentiable, with $c' = -s$ and $s' = c$.

(ii) Show that $c(x)^2 + s(x)^2 = 1$ for all x.

(iii) Show that $c(x) > 0$ for $0 \leqslant x \leqslant 1$, and that $c(1)^2 < \frac{1}{2}$. Hence, conclude that there is a unique $\sigma \in (0, 1)$ such that $c(\sigma) = s(\sigma)$.

(iv) Suppose that $F : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function such that $F'' = -F$ and $F(0) = F'(0) = 0$. Show that F is identically zero. (*Hint: you may want to use Taylor's theorem in the following form: for every $x > 0$ there is some $\theta_x \in (0, x)$ such that $F(x) = F(0) + xF'(0) + \frac{x^2}{2}F''(\theta_x)$.*)

(v) Show that $c(x + y) = c(x)c(y) - s(x)s(y)$, and give a similar formula for $s(x + y)$.

(vi) Show that $c(x + 8\sigma) = c(x)$, $s(x + 8\sigma) = s(x)$ for all x.

(vii) Show that $\sigma = \int_0^1 \frac{dx}{1+x^2}$.

3. Consider the following functions f_n:

$$\text{(i) } nx^n(x-1), \qquad \text{(ii) } \frac{x}{1+nx^2}, \qquad \text{(iii) } n^2xe^{-nx^2}.$$

In which cases does the sequence (f_n) converge uniformly on $[0, 1]$? In which cases is it true that $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 \lim_{n\to\infty} f_n$?

4. Let $f_n(t) = \frac{n}{n+t}$ for $t \geqslant 0$. Show that, for each $x > 0$, the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $[0, x]$. By considering $\int_0^x f_n$ deduce that $(1+\frac{x}{n})^n \to e^x$ as $n \to \infty$ for all $x \in \mathbb{R}$. (*You may use simple facts about* log *and* exp *without proof.*)

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