Length of a curve. Circumference of a circle.

Suppose we have a function $f(x)$, continuous on $[a,b]$ and differentiable on (a,b). We can draw its graph Γ, that is to say the set $\{(x,f(x)) : x \in [a,b]\} \subset \mathbb{R}^2$. What is the "length" of this curve?

For notational simplicity, let us take $a = 0$, $b = 1$.

Let $\mathcal{P}_n$ be the usual partition of $[0,1]$ into n parts, thus $x_i^{(n)} := i/n$ for $i = 0,\dots,n$. We can approximate Γ on each interval $[x_{i-1}^{(n)}, x_i^{(n)}]$ by the straight line segment joining $(x_{i-1}^{(n)}, f(x_{i-1}^{(n)}))$ to $(x_i^{(n)}, f(x_i^{(n)}))$. The total length of this straight-line approximation to Γ is easily computed (using Pythagoras' theorem) to be

$$L_n := \sum_{i=1}^{n} \sqrt{\frac{1}{n^2} + \big(f(\frac{i}{n}) - f(\frac{i-1}{n})\big)^2}.$$

By the mean-value theorem, there is some $\xi_i^{(n)} \in (x_{i-1}^{(n)}, x_i^{(n)})$ such that

$$f(\frac{i}{n}) - f(\frac{i-1}{n}) = \frac{1}{n} f'(\xi_i^{(n)}).$$

Substituting in gives

$$L_n = \sum_{i=1}^{n} \frac{1}{n}\sqrt{1 + f'(\xi_i^{(n)})^2}.$$

This is a Riemann sum, specifically

$$L_n = \Sigma(\sqrt{1+(f')^2}, \mathcal{P}_n, \vec{\xi}^{(n)}).$$

Assuming that $\sqrt{1+(f')^2}$ is integrable on $[0,1]$, it follows from Proposition 3.2 in the notes that

$$\lim_{n\to\infty} L_n = \int_0^1 \sqrt{1+(f')^2}. \tag{1}$$

Remark. If f' is integrable, then $\sqrt{1+(f')^2}$ is automatically integrable. We did not quite prove this in lectures, but it follows from Question 3 on the sheet of further questions.

The limit (1) obviously makes the following definition extremely natural.

Definition. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous, is differentiable on $(0,1)$ and that f' is integrable on $(0,1)$. Then we define the *length* $\ell(\Gamma)$ of the curve Γ to be $\int_0^1 \sqrt{1+(f')^2}$.

Let's try to compute the circumference C of a circle with unit radius. It is natural, by symmetry considerations, to define $C = 8\ell(\Gamma)$, where $\Gamma = \{(x, f(x)) : x \in [0, 1/\sqrt{2}]\}$, where $f(x) = \sqrt{1-x^2}$. By the chain rule, $f' = -x/\sqrt{1-x^2}$, and so

$$C = 8\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}.$$

Remark. It seems, at first sight, to be more natural to divide the circle into four, and look at the integral on $[0,1]$. However, f' becomes unbounded as $x \to 1$, so the length of this curve is not properly defined. One *can* make sense of it as an improper integral, an exercise we leave to the reader.

To evaluate C, we use the substitution rule. As many of you know, the substitution $x = \sin t$, $t \in [0, \pi/4]$, is a very sensible one. However, we have to be careful to avoid tautology, and must ask ourselves how we define the function $\sin t$. To this end, we refer to Exercise Sheet 3, Q2. There, we defined

$$c(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and

$$s(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We will make use of some of the properties established in that question, namely:

(i) The series for c, s converge for all x and that both functions are differentiable, with $c' = -s$ and $s' = c$.

(ii) $c(x)^2 + s(x)^2 = 1$ for all x.

(iii) $c(x) > 0$ for $0 \leqslant x \leqslant 1$, and there is a unique $\sigma \in (0,1)$ such that $c(\sigma) = s(\sigma)$.

Remark. Of course we "know" that $c = \cos$, $s = \sin$ and $\sigma = \pi/4$.

To evaluate $\ell(\Gamma)$, we make the substitution $x = s(t)$, $t \in [0, \sigma]$. That is, we intend to apply Proposition 4.2 from the notes (substitution) with $[a, b] = [0, 1/\sqrt{2}]$, $f(x) = 1/\sqrt{1 - x^2}$, $[c, d] = [0, \sigma]$, $\phi(t) = s(t)$. Let us check the conditions of that proposition:

- f is certainly continuous.
- ϕ is continuously differentiable, by (i).
- $\phi(c) = a$ is obvious from the definition of s.
- $\phi(d) = b$ follows from the fact that $c(\sigma) = s(\sigma)$ and (ii).
- To show that ϕ maps (c, d) to (a, b), we need to know that $s(t) \notin \{0, 1/\sqrt{2}\}$ for $0 < t < \sigma$. By (i) and (iii) we know that $s'(t) = c(t) > 0$ for $t > 0$, and so s is strictly increasing on $[0, \sigma]$. This is all we need.

Thus Proposition 4.2 may be applied, and we obtain

$$\ell(\Gamma) = \int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Now

$$f(\phi(t))\phi'(t) = \frac{1}{1 - s(t)^2} c(t) = 1,$$

by (ii). Finally, we see that $\ell(\Gamma) = \sigma = \pi/4$, and so the circumference of the unit circle is $8\sigma = 2\pi$.

Let us summarise what we have shown here. *Defining* π to be the 4σ, where σ is the smallest solution to $c(\sigma) = s(\sigma)$, with c and s given by series as above, we have *proven* that the circumference of a unit circle is 2π.

Exercise. Defining the *area* of the unit circle as $A := 4\int_0^1 \sqrt{1 - x^2}dx$, show that $A = 4\sigma$.

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