Problem Sheet 1

1. A company sells lottery scratch-cards for £1 each. 1% of cards win the grand prize of £50, a further 20% win a small prize of £2, and the rest win no prize at all. Estimate how many cards the company needs to sell to be 99% sure of making an overall profit. [$\Phi(2.3263) = 0.99$]

2. A list consists of 1000 non-negative numbers. The sum of the entries is 9000 and the sum of the squares of the entries of 91000. Let X represent an entry picked at random from the list. Find the mean of X, the mean of X^2, and the variance of X. Using Markov's inequality, show that the number of entries in the list greater than or equal to 50 is at most 180. What is the corresponding bound from applying Markov's inequality to the random variable X^2? What is the corresponding bound using Chebyshev's inequality?

3. For $n \geq 1$, let Y_n be uniform on $\{1, 2, \ldots, n\}$ (i.e. taking each value with probability $1/n$). Draw the distribution function of Y_n/n. Show that the sequence Y_n/n converges in distribution as $n \to \infty$. What is the limit?

4. Let $X_i, i \geq 1$, be i.i.d. uniform on $[0, 1]$. Let $M_n = \max\{X_1, \ldots, X_n\}$.
   (a) Show that $M_n \to 1$ in probability as $n \to \infty$.
   (b) Show that $n(1 - M_n)$ converges in distribution as $n \to \infty$. What is the limit?

5. (a) What is the distribution of the sum of n independent Poisson random variables each of mean 1? Use the central limit theorem to deduce that
$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2} \text{ as } n \to \infty.$$
   (b) Let $p \in (0, 1)$. What is the distribution of the sum of n independent Bernoulli random variables with parameter p? Let $0 \leq a < b \leq 1$. Use appropriate limit theorems to determine how the value of
$$\lim_{n\to\infty} \sum_{r\in\mathbb{N}:\ an \leq r < bn} \binom{n}{r} p^r (1-p)^{n-r}$$
   depends on a and b.

6. (a) Let X_n, $n \geq 1$, be a sequence of random variables defined on the same probability space. Show that if $X_n \to c$ in distribution, where c is a constant, then also $X_n \to c$ in probability.
   (b) Show that if $\mathbb{E}\,|X_n - X| \to 0$ as $n \to \infty$, then $X_n \to X$ in probability. Is the converse true?

7. A gambler makes a long sequence of bets against a rich friend. The gambler has initial capital C. On each round, a coin is tossed; if the coin comes up tails, he loses 30% of his current capital, but if the coin comes up heads, he instead wins 35% of his current capital.

   (a) Let C_n be the gambler's capital after n rounds. Write C_n as a product $CY_1Y_2\dots Y_n$ where Y_i are i.i.d. random variables. Find $\mathbb{E}\,C_n$.

   (b) Find the median of the distribution of C_{10} and compare it to $\mathbb{E}\,C_{10}$.

   (c) Consider $\log C_n$. What does the law of large numbers tell us about the behaviour of C_n as $n \to \infty$? How is this consistent with the behaviour of $\mathbb{E}\,C_n$?

8. Let $\mathbb{H}_n$ be the n-dimensional cube $[-1,1]^n$. For fixed $x \in \mathbb{R}$, show that the proportion of the volume of $\mathbb{H}_n$ within distance $(n/3)^{1/2} + x$ of the origin converges as $n \to \infty$, and find the limit. [Hint: Consider a random point whose n coordinates are i.i.d. with Uniform$[-1,1]$ distribution. If $A \subset \mathbb{H}_n$, then $\mathrm{vol}(A)/\mathrm{vol}(\mathbb{H}_n)$ is the probability that such a point falls in the set A. Let D_n represent the distance of such a point from the origin; apply an appropriate limit theorem to D_n^2.]

9. Let $Y_1, Y_2, \dots$ be i.i.d. and uniformly distributed on the set $\{1, 2, \dots, n\}$. Define $X^{(n)} = \min\{k \geq 1 \colon Y_k = Y_j \text{ for some } j < k\}$, the first time that we see a repetition in the sequence Y_i. (Interpret the case $n = 365$). Prove that $X^{(n)}/\sqrt{n}$ converges in distribution to a limit with distribution function $F(x) = 1 - \exp(-x^2/2)$ for $x > 0$.
   [Hint: Observe that
   $$\mathbb{P}\left(X^{(n)} > m\right) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$
   You may find it useful to use bounds such as $-h - h^2 < \log(1-h) < -h$ for sufficiently small positive h.]

Additional problems:

10. Let $X_i, i \geq 1$, be i.i.d. random variables with $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = 1/2$.

    (a) Define $S_n = \sum_{i=1}^n X_i 2^{-i}$. What is the distribution of S_n? Show that the sequence S_n converges almost surely as $n \to \infty$ (hint: Cauchy sequences converge). What is the distribution of the limit?

    (b) Define $R_n = \sum_{i=1}^n 2X_i 3^{-i}$. Show again that the sequence R_n converges almost surely. Is the limit a discrete random variable? Is it a continuous random variable? (Consider its expansion in base 3).

11. Let A_n be the median of $2n+1$ i.i.d. random variables which are uniform on $[0,1]$. Find the probability density function of A_n. (Hint: consider the probability that A_n lies in a small interval $(x, x+dx)$). How does the density at the point $\left(\frac{1}{2} + \frac{a}{\sqrt{n}}\right)$ behave as $n \to \infty$? (Stirling's formula may be useful). Deduce a convergence in distribution result for the median (appropriately rescaled) as $n \to \infty$ (feel free to argue informally if you like!).

Problem Sheet 2

1. (a) Let X and Y be independent standard normal random variables. Define R and Θ by $X = R\cos\Theta$, $Y = R\sin\Theta$. Find the joint distribution of R and Θ.

   (b) Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal random variables. Let A be an orthogonal $n \times n$ matrix. Find the joint distribution of $W_1, W_2, \ldots, W_n$ where $\mathbf{W} = A\mathbf{Z}$. Explain the link to part (a).

2. The Gamma(r, λ) distribution has density

$$f_{r,\lambda}(x) = \frac{1}{\Gamma(r)}\lambda^r x^{r-1} e^{-\lambda x}$$

   on $\mathbb{R}_+$. Here r is called the *shape* parameter and λ is called the *rate* parameter.

   Use moment generating functions to show that the sum of two independent Gamma-distributed random variables with the same rate parameter is also Gamma-distributed. (What does this say about sums of exponential random variables?)

3. A student makes repeated attempts to solve a problem. Suppose the ith attempt takes time X_i, where X_i are i.i.d. exponential random variables with parameter λ. Each attempt is successful with probability p (independently for each attempt, and independently of the durations). Use moment generating functions to show that the distribution of the total time before the problem is solved has an exponential distribution, and find its paramter.

4. (a) Let X, Y and U be independent random variables, where X and Y have moment generating functions $M_X(t)$ and $M_Y(t)$, and where U has the uniform distribution on $[0, 1]$.

   Find random variables which are functions of X, Y and U and which have the following moment generating functions: (i) $M_X(t)M_Y(t)$; (ii) $e^{bt}M_X(at)$; (iii) $\int_0^1 M_X(tu)du$; (iv) $[M_X(t) + M_Y(t)]/2$.

   (b) Using characteristic functions or otherwise, find $\mathbb{E}\cos(tX)$ and $\mathbb{E}\sin(tX)$ when X has exponential distribution with parameter λ.

   (c) Which random variables X have a real-valued characteristic function?

5. Suppose X has Gamma$(2, \lambda)$ distribution, and the conditional distribution of Y given $X = x$ is uniform on $(0, x)$.

   Find the joint density function of X and Y, the marginal density function of Y, and the conditional density function of X given $Y = y$? How would you describe the distribution of X given $Y = y$? Use this to describe the joint distribution of Y and $X - Y$.

6. Random variables X and Y have joint density $f(x, y)$. Let $Z = Y/X$. Show that Z has density

$$f_Z(z) = \int_{-\infty}^{\infty} |x|f(x, xz)dx.$$

   Suppose now that X and Y are independent standard normal random variables. Show that Z has density

$$f_Z(z) = \frac{1}{\pi(1 + z^2)}, \ -\infty < z < \infty.$$

7. The distribution of the heights of husband-wife pairs in a particular population is modelled by a bivariate normal distribution. The mean height of the women is 165cm and the mean height of the men is 175cm. The standard deviation is 6cm for women and 8cm for men. The correlation of height between husbands and wives is 0.5.

   Let X be the height of a typical wife and Y the height of her husband. Show how Y can be represented as a sum of a term which is a multiple of X and a term which is independent of X. Hence or otherwise:

   (a) Given that a woman has height 168cm, find the expected height of her husband.

   (b) Given that a woman has height 168cm, what is the probability that her husband is above average height?

   (c) What is the probability that a randomly chosen man is taller than a randomly chosen woman?

   (d) What is the probability that a randomly chosen man is taller than his wife?

8. (a) Let X and Y be independent standard normal random variables. Use question 1(a) to show that for a constant $c > 0$,

$$\mathbb{P}(X > 0, Y > -cX) = \frac{1}{4} + \frac{\tan^{-1}(c)}{2\pi}.$$

   (b) Two candidates contest a close election. Each of the n voters votes independently with probability $1/2$ each way. Fix $\alpha \in (0, 1)$. Show that, for large n, the probability that the candidate leading after αn votes have been counted is the eventual winner is approximately

$$\frac{1}{2} + \frac{\sin^{-1}(\sqrt{\alpha})}{\pi}.$$

   [Hint: let S_m be the difference between the vote totals of the two candidates when m votes have been counted. What is the approximate distribution of $S_{\alpha n}$ (when appropriately rescaled)? What is the approximate distribution of $S_n - S_{\alpha n}$ (when appropriately rescaled)? What about their joint distribution? Finally, notice $\sin^{-1}(\sqrt{\alpha}) = \tan^{-1}\sqrt{\alpha/(1-\alpha)}$.]

Additional problems:

9. Let U, V and W be i.i.d. random variables with uniform distribution on $[0, 1]$. Find the distribution of $(UV)^W$.

10. Use characteristic functions to prove the identity

$$\frac{\sin t}{t} = \prod_{n=1}^{\infty} \cos\left(\frac{t}{2^n}\right).$$

   [Hint: consider the c.f. of a uniform distribution, and of a distribution taking only two values.]

Problem Sheet 3

1. Find the communicating classes of the Markov chains with the following transition matrices on the state space $\{1, 2, 3, 4, 5\}$, and in each case determine which classes are closed:

$$\text{(i)} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}.$$

If X is a chain with the transition matrix in (ii), find the distribution of X_1 when X_0 has the uniform distribution on $\{1, 2, 3, 4, 5\}$, and find $P(X_2 = 3 \,|\, X_0 = 1)$.

2. N black balls and N white balls are distributed between two urns, numbered 1 and 2, so that each urn contains N balls. At each step, one ball is chosen at random from each urn and the two chosen balls are exchanged. Let X_n be the number of white balls in urn 1 after n steps. Find the transition matrix for the Markov chain X.

3. A die is "fixed" so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probablity $1/5$. If the first score is 6, what is the probability that the nth score is 6 and what is the probability that the nth score is 1? [*Hint: you can simplify things by selecting an appropriate state-space; do you really need a* 6*-state chain to answer the question?*]

4. Let $X_n, n \geq 1$, be i.i.d. taking value 1 with probability p and -1 with probability $1 - p$, where $p \in (0, 1)$. In each of the following cases, decide whether $Y_n, n \geq 1$, is a Markov chain. If so, find its transition probabilities.

   (a) $Y_n = X_n$.
   (b) $Y_n = S_n$ where $S_n = X_1 + X_2 + \cdots + X_n$.
   (c) $Y_n = M_n$ where $M_n = \max(0, S_1, S_2, \ldots, S_n)$.
   (d) $Y_n = M_n - S_n$.
   (e) $Y_n = X_n X_{n+1}$.

5. Let C be a communicating class of a Markov chain. Prove the following statements:

   (a) Either all states in C are recurrent, or all are transient. (So we may refer to the whole class as transient or recurrent.) [*Hint: use the criterion for recurrence of a state i in terms of $\sum p_{ii}^{(n)}$ to show that if i is recurrent and $i \leftrightarrow j$ then also j is recurrent.*]
   (b) If C is recurrent then C is closed.
   (c) If C is finite and closed, then C is recurrent.

6. A gambler has £8 and wants to increase it to £10 in a hurry. He can repeatedly stake money on the toss of a fair coin; when the coin comes down tails, he loses his stake, and when the coin comes down heads, he wins an amount equal to his stake, and his stake is returned.

   He decides to use a strategy in which he stakes all his money if he has less than £5, and otherwise stakes just enough to increase his capital to £10 if he wins. For example, he will stake £2 on the first coin toss, and afterwards will have either £6 or £10.

(a) Let £X_n be his capital after the nth coin toss. Show how to describe the sequence $(X_n, n \geq 0)$ as a Markov chain.

(b) Find the expected number of coin tosses until he either reaches £10 or loses all his money.

(c) Show that he reaches £10 with probability 4/5.

(d) Show that the probability that he wins the first coin toss, given that he eventually reaches £10, is 5/8. Extend this to describe the distribution of the whole sequence $X_0, X_1, X_2, \ldots$ conditional on the event that he reaches £10.

(e) In a similar way, let $(X_n, n \geq 0)$ be a Markov chain on $\mathbb{N}$ with $p_{i,i+1} = p = 1 - p_{i,i-1}$ for $i \geq 1$, and $p_{0,0} = 1$. Let $p > 1/2$ so that the process has an upward bias. Start at $X_0 = j > 0$. In lectures we showed that the probability of absorption at 0 is $\left(\frac{1-p}{p}\right)^j$. Describe the distribution of $(X_n, n \geq 0)$ conditional on the event of being absorbed at 0.

7. A Markov chain with state space $\{0, 1, 2, \ldots\}$ is called a "birth-and-death chain" if the only non-zero transitions from state i are to states $i-1$ and $i+1$.

   Consider a general birth-and-death chain and write $p_i = p_{i,i+1}$ and $q_i = p_{i,i-1} = 1 - p_i$. Assume that p_i and q_i are positive for all $i \geq 1$.

   Let h_i be the probability of reaching 0 starting from i, and write $u_i = h_{i-1} - h_i$.

   (a) Show that $p_i h_i + q_i h_i = h_i = p_i h_{i+1} + q_i h_{i-1}$, and hence that $u_{i+1} = \dfrac{q_i}{p_i} u_i$.

   (b) Define $\gamma_i = \dfrac{q_1}{p_1} \dfrac{q_2}{p_2} \cdots \dfrac{q_{i-1}}{p_{i-1}}$.
   Write u_i in terms of γ_i and u_1, and then h_i in terms of $\gamma_1, \ldots, \gamma_i$ and u_1.

   (c) The equations for $h_1, h_2, \ldots$ may have multiple solutions. Which solution gives the true hitting probabilities? Hence find the value of u_1, and deduce that the chain is transient if and only if $\sum_{i=1}^{\infty} \gamma_i$ is finite.

   (d) Consider the case where
   $$p_i = \left(\frac{i+1}{i}\right)^2 q_i.$$
   Show that if $X_0 = 1$, then $\mathbb{P}(X_n \geq 1 \text{ for all } n \geq 1) = 6/\pi^2$.

Additional problems:

8. Suppose P is an irreducible transition matrix, with period d. Consider the transition matrix P^k. In terms of d and k, how many communicating classes does P^k have, and what is the period of each state?

9. Consider a random walk on a cycle of size M; that is, a Markov chain with state space $\{0, 1, \ldots, M-1\}$ and transition probabilities
   $$p_{ij} = \begin{cases} 1/2 & \text{if } j \equiv i+1 \mod M \\ 1/2 & \text{if } j \equiv i-1 \mod M \\ 0 & \text{otherwise} \end{cases}.$$
   The walk starts at 0. What is the distribution of the last site to be reached by the chain?

10. Continuing question 4, let $Y_n = |S_n|$. Does this give a Markov chain?

Problem Sheet 4

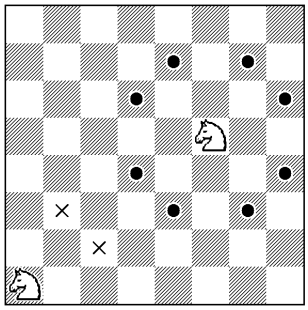
1. Find the stationary distributions of the following transition matrices. In each case describe the limiting behaviour of $p_{12}^{(n)}$ as $n \to \infty$.

$$\text{(i)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}\,(X_n \text{ is a multiple of } 11)\,.$$

3. A knight performs a random walk on a chessboard, making each possible move with equal probability at each step. If the knight starts in the bottom left corner, how many moves on average will it take to return there? (The knight's possible moves from two different positions are shown in the picture.) [*Hint: consider the "random walk on a graph" from lectures, in which the equilibrium probability of a vertex is proportional to its number of neighbours.*]
Let p_n be the probability that the knight is back in the same corner after n steps. Describe the behaviour of p_n as $n \to \infty$.



4. A frog jumps on an infinite ladder. At each jump, with probability $1-p$ he jumps up one step, while with probability p he slips off and falls all the way to the bottom.

   Represent the frog's height on the ladder as a Markov chain; show that the stationary distribution is geometric, and find its parameter. Use the ergodic theorem to obtain a precise statement about the long-run proportion of time for which the frog is on the first step above the bottom of the ladder.

   If the frog has just fallen to the bottom, on average how many jumps will it take before he next reaches step k? (One approach: consider the mean return time from k to itself.)

5. A professor owns N umbrellas. She walks to work each morning and back from work each evening. On each walk, if she has an umbrella available, she carries it if and only if it is raining. If it is raining and there is no umbrella available, she walks anyway and gets wet. Suppose it is raining independently on each walk with probability p. What is the long-run proportion of walks on which she gets wet? [*Hint: define a Markov chain to represent the number of umbrellas at home at the end of each day. If π is the stationary distribution, then for $2 \le i \le N-1$ the relation $\pi_i = \sum_j p_{ji}\pi_j$ should give $\pi_i = \big(\pi_{i-1} + \pi_{i+1}\big)/2$. Add in the relations for $i = 0$ and $i = N$ to find π.*]

6. Starting from some fixed time, requests at a web server arrive at an average rate of 2 per second, according to a Poisson process. Find: (a) the probability that the first request arrives within 2 seconds. (b) the distribution of the number of requests arriving within the first 5 seconds. (c) the distribution of the arrival time of the nth request; give its mean and its variance (these should not require much calculation!). (d) the approximate probability that more than 7250 requests arrive within the first hour.

7. Arrivals of the Number 2 bus form a Poisson process of rate 2 per hour, and arrivals of the Number 7 bus form a Poisson process of rate 7 buses per hour, independently.

   (a) What is the probability that exactly three buses pass by in an hour?

   (b) What is the probability that the first Number 2 bus arrives before the first Number 7 bus?

   (c) When the maintenance depot goes on strike, each bus breaks down independently with probability half before reaching my stop. In that case, what is the probability that I wait for 30 minutes without seeing a single bus?

8. Let N_t be a Poisson process of rate λ. What is $\mathbb{P}(N_t = 1)$? What is $\mathbb{P}(N_s = 1 \mid N_t = 1)$ for $0 < s < t$? (Note carefully which properties of the Poisson process you are using.) Hence find the distribution of the time of the first point of the process, conditional on the event that exactly one point occurs in the interval $[0, t]$.

9. Let N_t be a Poisson process of rate λ. Define $X_n = N_n - n$ for $n = 0, 1, 2, \ldots$.

   Explain why X_n is a Markov chain and give its transition probabilities.

   Use the strong law of large numbers to show that the chain is transient if $\lambda \neq 1$.

   If $\lambda = 1$, is the chain transient? null recurrent? positive recurrent? (Stirling's formula and the criterion for recurrence in terms of the sequence $p_{00}^{(n)}$ may help.)

10. Let $T > 1$. We observe a Poisson process of rate 1 on the time interval $(0, T)$. Each time a point occurs, we may decide to stop. Our goal is to stop at the last point which occurs before time T; if so, we win, and otherwise – i.e. if we never stop, or if we stop at some time t but another point occurs in (t, T) – we lose.

    Find the best strategy you can for playing this game. What is its probability of winning? Can you show that it's optimal? Feel free to argue informally (but convincingly!).