Metric spaces and complex analysis

Mathematical Institute, University of Oxford
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Problem Sheet 1

1. Let $X = \mathbb{N}^{\mathbb{N}}$ be the set of all sequences of natural numbers ($\mathbb{N} = \{1, 2, \ldots\}$). If $(x_n), (y_n)$ are elements of X set

$$d((x_n), (y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}},$$

(where the right-hand side is to be interpreted as zero when $x_n = y_n$ for all n). Show that (X, d) is a metric space.

2. Let (X, d) be a metric space and let $f\colon X \to \mathbb{R}$ is a continuous function (where $\mathbb{R}$ is a metric space using the standard distance $d(a, b) = |a - b|$ for all $a, b \in \mathbb{R}$). Show that if $a \in X$ is such that $f(a) \neq 0$, then there is an $r > 0$ such that the function $1/f$ is defined and is continuous on $B(a, r)$.

3. Let X, Y and Z be metric spaces and equip $Y \times Z$ with the d_2 metric, so that

$$d((y_1, z_1), (y_2, z_2)) = \sqrt{d(y_1, y_2)^2 + d(z_1, z_2)^2}, \quad \forall y_1, y_2 \in Y, z_1, z_2 \in Z.$$

If $F\colon X \to Y \times Z$ is a function and we write $F(x) = (f_1(x), f_2(x))$, show that F is continuous if and only if f_1 and f_2 are continuous.

4. Show that the functions of addition and multiplication of real numbers are continuous. Deduce that if $f, g\colon X \to \mathbb{R}$ are continuous functions, so are $f + g$ and $f.g$.

5. Let ℓ_2 denote the space of real-valued sequences which are square summable, that is, sequences (x_n) where $x_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} x_N^2 < \infty$. Show that ℓ_2 equipped with the function $\|(x_k)\|_2 = (\sum_{k=1}^{\infty} x_k^2)^{1/2}$ is a normed $\mathbb{R}$-vector space. (*Thus you need to check that ℓ_2 is a vector space and that $\|.\|_2$ defines a norm on it.*)

The function d_2 given by

$$d_2((x_n), (y_n)) = \left(\sum_{n=1}^{\infty} (x_n - y_n)^2\right)^{1/2},$$

is the metric associated to the norm $\|.\|_2$.

Consider also the function $\|.\|_\infty$ given by

$$\|(x_k)\|_\infty = \sup_{k \in \mathbb{N}} |x_k|.$$

Say briefly why this is well-defined. Is this a norm? Are the norms $\|.\|_2$ and $\|x\|_\infty$ equivalent, that is, do there exist constants C_1, C_2 such that $\|(x_k)\|_\infty \leq C_1 \|(x_k)\|_2$ and $\|(x_k)\|_2 \leq C_2 \|(x_k)\|_\infty$ for all $(x_k) \in \ell_2$?

6. Say which of the following subsets of $\mathbb{R}^2$ are open.

$$B(0, 1); \quad [0, 1] \times [0, 1]; \quad (0, 1) \times (0, 1), \quad (0, 1) \times \{0\}.$$

7. Let (X, d) be a metric space. Show that the closure of the union of two subsets A, B of X is the union of the closures of A and B, that is $\overline{A \cup B} = \bar{A} \cup \bar{B}$. Is $\bar{A} \cap \bar{B} = \overline{A \cap B}$?

8. Let $A \subseteq [0, 1]$ be the set of real numbers which can be expressed in the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0, 1\}$. By considering the sets $A_N = \{x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, a_k \in \{0, 1\}, \forall k \leq N\}$ or otherwise, find the closure of A. [*You may assume that any $y \in [0, 1]$ can be written in the form $\sum_{k=1}^{\infty} \frac{b_k}{3^k}$ where $b_k \in \{0, 1, 2\}$.*]

9 (*Optional*). Let $\mathbb{N}$ be the set of natural numbers. Let $\mathcal{A}$ be the set of arithmetic progressions in $\mathbb{N}$, that is, subsets of the form $A_{a,b} = \{a + b(n - 1) : n \in \mathbb{N}\}$ where a, b are fixed positive integers. Let $\mathcal{U}$ be the collection of subsets of $\mathbb{N}$ which are (arbitrary) unions of elements of $\mathcal{A}$, *i.e.* unions of arithmetic progressions. Show that $\mathcal{U}$ is a topology on $\mathbb{N}$. Deduce that there are infinitely many primes.
[*It may help to consider which subsets of $\mathbb{N}$ are both open and closed.*]