Metric spaces and complex analysis

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Problem Sheet 2

1. Let $\ell_2 = \{(x_k)_{k\geq 1} : \sum_{k=1}^{\infty} x_k^2 < \infty\}$ be the normed vector space of square-summable sequences (equipped with norm $\|.\|_2$ given by $\|(x_n)\|_2 = \left(\sum_{k=0}^{\infty} x_k^2\right)\big)^{1/2}$ as in problem sheet 1). Show that ℓ_2 is complete.
[*You should try to mimic the proof that the space of bounded real-valued functions on a set is complete.*]

2. Suppose that X is a complete metric space and $A \subseteq X$ is a subspace. Show that the subspace A equipped with the induced metric "knows about its closure" in the following sense: Let $\mathcal{C}_A$ denote the set of all Cauchy sequences in A. Equip $\mathcal{C}_A$ with a relation $\sim$ as follows: Say that $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$ as $n \to \infty$. Check that this is an equivalence relation and let $\bar{\mathcal{C}}_A$ denote its equivalence classes. Show that there is a natural bijection between $\bar{\mathcal{C}}_A$ and $\bar{A} \subseteq X$.

3. Let (X, d) be a metric space and let $T\colon X \to X$ be a contraction.
   - *i*) Give an example where X is incomplete, T is a contraction, and such that T has no fixed point.
   - *ii*) Give an example where X is complete, and the map $T\colon X \to X$ is such that $d(T(x), T(y)) < d(x, y)$ but T does not have any fixed point.
   - *iii*) Give an example where X is complete and the map T is not a contraction, but for which there is some $m \in \mathbb{Z}_{>0}$ with T^m a contraction.

4. Let (X, d) be a metric space.
   - *i*) Show directly from the definition that X is connected if and only if every integer-valued continuous function on X is constant.
   - *ii*) Now suppose that $X = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Show that X is connected. By considering the function $f(x, y)/x$ show that there are precisely two continuous functions $f\colon X \to \mathbb{R}$ satisfying $f(x, y)^2 = x^2$ for all $(x, y) \in X$.
   - *iii*) How many continuous functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ are there satisfying $g(x, y)^2 = x^2$ for all $(x, y) \in \mathbb{R}^2$?

5.
   - *i*) Show that if U is an open subset of $\mathbb{R}$ and $c \in U$ then $U \backslash \{c\}$ is not connected.
   - *ii*) Show that if U is a connected open subset of $\mathbb{R}^2$ and $c \in U$ then $U \backslash \{c\}$ is connected.
   - *iii*) Show that there are no continuous injective maps from $\mathbb{R}^2$ to $\mathbb{R}$.

6. Show that a subset $X \subseteq \mathbb{R}^n$ is compact if and only if every continuous function $f\colon X \to \mathbb{R}$ is bounded.

7. Show that any norm on $\mathbb{R}^n$ is equivalent to the $\|.\|_1$-norm, that is, show that if $\|.\|$ is any norm on $\mathbb{R}^n$ then there are positive constants C, D such that $\|v\|_1 \leq C\|v\|$ and $\|v\| \leq D\|v\|_1$ for all $v \in \mathbb{R}^n$.

   Deduce that any two norms on a finite dimensional real vector space are equivalent.
Optional: Why does this imply that any finite dimensional subspace of a normed vector space must be closed?
Hint: the set $S = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$ is closed and bounded as a subset of the normed vector space $(\mathbb{R}^n, \|.\|_1)$, hence it is compact, and hence any continuous function on it must attain its minimum.

8. Let M be a metric space and $X_1, X_2, \ldots$ an infinite collection of subsets of M. For each of the following statements, give a proof or counterexample.
   - *i*) If $X_1, X_2, \ldots, X_k$ are compact then $X_1 \cup X_2 \cup \ldots \cup X_k$ is compact.
   - *ii*) If $X_1, X_2, \ldots, X_k$ are connected then $X_1 \cap X_2 \cap \ldots \cap X_k$ is connected.
   - *iii*) If $X_1, X_2, \ldots$ are compact then $\bigcup_{k\geq 1} X_k$ is compact.
   - *iv*) If $X_1, X_2, \ldots$ are connected and $X_j \cap X_{j+1} \neq \emptyset$ then $\bigcup_{k\geq 1} X_k$ is connected.

9 (*Optional*). : Let $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^n : \sum_{k=0}^{n} x_k^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$. It is a metric space with restriction of the standard d_2 metric on $\mathbb{R}^{n+1}$. Assuming that the isometries of $\mathbb{R}^{n+1}$ which fix the origin are given by the group of orthogonal matrices $\mathbf{O}_{n+1}$, show that the group of isometries of S^n is exactly the group $\mathbf{O}_{n+1}$.