6. Subspaces of metric spaces

If (X, d) is a metric space, then as we noted before, any subset $Y \subseteq X$ is automatically also a metric space since the distance function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ restricts to a distance function on Y. The set Y thus has a topology given by this metric. In this section we show that this topology is easy to describe in terms of the topology on X. The key to this description is the simple observation that the open balls in Y are just the intersection of the open balls in X with Y. For clarity, for $y \in Y \subseteq X$ we will write

$$B_Y(y, r) = \{z \in Y : d(z, y) < r\}$$

for the open ball about y of radius r in Y and

$$B_X(y, r) = \{x \in X : d(x, y) < r\}$$

for the open ball of radius r about y in X. Thus $B_Y(y, r) = Y \cap B_X(y, r)$.

Lemma 6.1. *If (X, d) is a metric space and $Y \subseteq X$ then a subset $U \subseteq Y$ is an open subset of Y if and only if there is an open subset V of X such that $U = V \cap Y$. Similarly a subset $Z \subseteq Y$ is a closed subset of Y if and only if there is a closed subset F of X such that $Z = F \cap Y$.*

Proof. If $U = Y \cap V$ where V is open in X and $y \in U$ then there is a $\delta > 0$ such that $B_X(y, \delta) \subseteq V$. But then $B_Y(y, \delta) = B_X(y, \delta) \cap Y \subseteq V \cap Y = U$ and so U is a neighbourbood of each of its points as required. On the other hand, if U is an open subset of Y then for each $y \in U$ we may pick an open ball $B_Y(y, \delta_y) \subseteq U$. It follows that $U = \bigcup_{y \in U} B_Y(y, \delta_y)$. But then if we set $V = \bigcup_{y \in U} B_X(y, \delta_y)$ it is immediate that V is open in X and $V \cap Y = U$ as required.

The corresponding result for closed sets follows readily: F is closed in Y if and only if $Y \backslash F$ is open in Y which by the above happens if and only if it equals $Y \cap V$ for some open set in X. But this is equivalent to $F = Y \cap V^c$, the intersection of Y with the closed set V^c. □

Remark 6.2. The lemma shows that the topology on X determines the topology on the subspace $Y \subseteq X$ directly. It is easy to see that if $(X, \mathcal{T})$ is an abstract topological space and $Y \subseteq X$ then the collection $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y which is called the *subspace topology*.

Remark 6.3. It is important here to note that the property of being open or closed is a relative one – it depends on which metric space you are working in. Thus for example if (X, d) is a metric space and $Y \subseteq X$ then Y is always open viewed as a subset of itself (since the whole space is always an open subset) but it of course need not be an open subset of X! For example, $[0, 1]$ is not open in $\mathbb{R}$ but it is an open subset of itself.

Example 6.4. Let's consider a more interesting example: Let $X = \mathbb{R}$ and let $Y = [0, 1] \cup [2, 3]$. As a subset of Y the set $[0, 1]$ is both open and closed. To see that it is open, note that if $x \in [0, 1]$ then

$$\begin{aligned} B_Y(x, 1/2) = B_{\mathbb{R}}(x, 1/2) \cap Y &= (x - \frac{1}{2}, x + \frac{1}{2}) \cap ([0, 1] \cup [2, 3]) \\ &= (x - \frac{1}{2}, x + \frac{1}{2}) \cap [0, 1] \subset [0, 1], \end{aligned}$$

Similarly we see that $B_Y(x, 1/2) \subseteq [2, 3]$ if $x \in [2, 3]$ so that $[2, 3]$ is also open in Y. It follows $[0, 1]$ is both open and closed in Y (as is $[2, 3]$).

7. Homeomorphisms and isometries

If (X, d) and (Y, d) are metric spaces it is natural to ask when we wish to consider X and Y equivalent. There is more than one way to answer this question – the first, perhaps most obvious one, is the following:

Definition 7.1. A function $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is said to be an *isometry* if

$$d_Y(f(x), f(y)) = d_X(x, y) \quad \forall x, y \in X$$

An isometry is automatically injective. If there is a surjective (and hence bijective) isometry between two metric spaces X and Y we say that X and Y are *isometric*.

Example 7.2. Let $X = \mathbb{R}^2$ (equipped with the Euclidean metric[13] d_2). The collection of all bijective isometries from X to itself forms a group, the *isometry group* of the plane. Clearly the translations $t_v : \mathbb{R}^2 \to \mathbb{R}^2$ are isometries, where $v \in \mathbb{R}^2$ and $t_v(x) = x + v$. Similarly, if $A \in \mathrm{Mat}_2(\mathbb{R})$ is an orthogonal matrix, so that $A^t A = I$, then $x \mapsto Ax$ is an isometry: since $d_2(Ax, Ay) = \|A(x) - A(y)\| = \|A(x - y)\|$ it is enough to check that $\|Ax\| = \|x\|$, but this is clear since $\|Ax\|^2 = (Ax).(Ax) = xA^t Ax = x^t Ix = \|x\|$.

In fact these two kinds of isometries generate the full group of isometries. If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is any isometry, let $v = T(0)$. Then $T_1 = t_{-v} \circ T$ is an isometry which fixes the origin. Thus it remains to show that any isometry which fixes the origin is in fact linear. But you showed in Prelims Geometry that any such isometry of $\mathbb{R}^n$ must preserve the inner product (because it preserves the norm and you can express the inner product in terms of the norm). It follows such an isometry takes an orthonormal basis to an orthonormal basis, from which linearity readily follows. (Note that this argument works just as well in $\mathbb{R}^n$.)

Example 7.3. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ be the n-sphere (so S^1 is a circle and S^2 is the usual sphere). Clearly $O_{n+1}(\mathbb{R})$ acts by isometries on S^n. In fact you can show that $\mathrm{Isom}(S^n) = O_{n+1}(\mathbb{R})$. To prove this one needs to show that any isometry of S^n extends to an isometry of $\mathbb{R}^{n+1}$ which fixes the origin.

We have already seen that on $\mathbb{R}^n$ the metrics d_1, d_2, d_∞, although different, induce the same notion of convergence and continuity[14] . The notion of isometry is thus in some sense too rigid a notion of equivalence if these are the notions we are primarily interested in. A weaker, but often more useful, notion of equivalence is the following:

Definition 7.4. Let $f: X \to Y$ be a continuous function between metric spaces X and Y. We say that f is a *homeomorphism* if there is a continuous function $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If there is a homeomorphism between two metric spaces X and Y we say they are *homeomorphic*.

Remark 7.5. Note that the definition implies that f is bijective as a map of sets but it is *not* true in general[15] that a continuous bijection is necessarily a homeomorphism. To see this, consider the spaces $X = [0,1) \cup [2,3]$ and $Y = [0,2]$. Then the function $f: X \to Y$ given by

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ x-1, & \text{if } x \in [2,3] \end{cases}$$

is a bijection and is clearly continuous. Its inverse $g: Y \to X$ is however not continuous at 1 – the one-sided limits of g as x tends to 1 from above and below are 1 and 2 respectively.

Example 7.6. The closed disk $\bar{B}(0,1)$ of radius 1 in $\mathbb{R}^2$ is homoemorphic to the square $[-1,1] \times [-1,1]$. The easiest way to see this is inscribe the disk in the square and stretch the disk radially out to the square. One can write explicit formulas for this in the four quarters of the disk given by the lines $x \pm y = 0$ to check this does indeed give a homeomorphism.

The open interval $(0,1)$ is homeomorphic to $\mathbb{R}$: a homeomorphism between them is given by the function $x \mapsto \tan(\pi.(x - 1/2))$, which has inverse $y \mapsto \frac{1}{\pi}\arctan(y) + \frac{1}{2}$.

8. Completeness

One of the important notions in Prelims analysis was that of a Cauchy sequence. This is a notion, like convergence, which makes sense in any metric space.

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[13]Unless it is explicitly stated otherwise, we will always take $\mathbb{R}^n$ to be a metric space equipped with the d_2 metric.

[14]There is actually a slightly subtle point here – to know that $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_2)$ are not isometric we would need to show that there is no bijective map $\alpha: \mathbb{R}^n \to \mathbb{R}^n$ such that $d_2(\alpha(x), \alpha(y)) = d_1(x, y)$ for all $x, y \in \mathbb{R}^n$.

[15]This is unlike the examples you have seen in algebra – the inverse of a bijective linear map is automatically linear, and the inverse of a bijective group homomorphism is automatically a homomorphism. Similarly, the inverse of a bijective isometry is also an isometry.

Definition 8.1. Let (X,d) be a metric space. A sequence (x_n) in X is said to be a *Cauchy sequence* if, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

The following lemma establishes basic properties of Cauchy sequences in an arbitrary metric space which you saw before for real sequences.

Lemma 8.2. *Let (X,d) be a metric space.*

(1) *If (x_n) is a convergent sequence then it is Cauchy.*
(2) *Any Cauchy sequence is bounded.*

Proof. Suppose that $x_n \to \ell$ as $n \to \infty$ and $\epsilon > 0$ is given. Then there is an $N \in \mathbb{N}$ such that $d(x_n, \ell) < \epsilon/2$ for all $n \geq N$. It follows that if $n, m \geq N$ we have

$$d(x_n, x_m) \leq d(x_n, \ell) + d(\ell, x_m) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so that (x_n) is a Cauchy sequence as required.

If (x_n) is a Cauchy sequence, then taking $\epsilon = 1$ in the definition, we see that there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. It follows that if we set

$$M = \max\{1, d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N)\}$$

then for all $n \in \mathbb{N}$ we have $x_n \in B(x_N, M)$ so that (x_n) is bounded as required. □

Part (1) of the lemma motivates the following definition:

Definition 8.3. A metric space (X,d) is said to be *complete* if every Cauchy sequence in X converges.

Example 8.4. One of the main results in Analysis I was that $\mathbb{R}$ is complete, and it is easy to deduce from this that $\mathbb{R}^n$ is complete also (since a sequence in $\mathbb{R}^n$ converges if and only if each of its coordinates converge).

On the other hand, consider the metric space $(0,1]$: The sequence $(1/n)$ converges in $\mathbb{R}$ (to 0) so the sequence is Cauchy in $\mathbb{R}$ and hence also in $(0,1]$, however it does not converge in $(0,1]$.

The previous example suggests a connection between completeness and closed sets. One precise statement of this form is the following:

Lemma 8.5. *Let (X,d) be a complete metric space and let $Y \subseteq X$. Then Y is complete if and only if Y is a closed subset of X.*

Proof. Note that if (x_n) is a Cauchy sequence in Y then it is certainly a Cauchy sequence in X. Since X is complete, (x_n) converges in X, say $x_n \to a$ as $n \to \infty$. Thus (x_n) converges in Y precisely when $a \in Y$. It follows that Y is complete if and only if it contains the limits of all sequences (x_n) in Y which converge in X. But Lemma 5.30 shows that the set of limits of all sequences in Y is exactly $\bar{Y}$, hence Y is complete if and only if $\bar{Y} \subseteq Y$, that is, if and only if Y is closed.

□

Another useful consequence of completeness is that it guarantees certain intersections of closed sets are non-empty:

Lemma 8.6. *Let (X,d) be a complete metric space and suppose that $D_1 \supseteq D_2 \supseteq \ldots$ form a nested sequence of non-empty closed sets in X with the property that diam$(D_k) \to 0$ as $k \to \infty$. Then there is a unique point $w \in X$ such that $w \in D_k$ for all $k \geq 1$.*

Proof. For each k pick $z_k \in D_k$. Then since the D_k are nested, $z_k \in D_l$ for all $k \geq l$, and hence the assumption on the diameters ensures that (z_k) is a Cauchy sequence. Let $w \in X$ be its limit. Since D_k is closed and contains the subsequence $(z_{n+k})_{n\geq 0}$ it follows $w \in D_k$ for each $k \geq 1$. To see that w is unique, suppose that $w' \in D_k$ for all k. Then $d(w, w') \leq \mathrm{diam}(D_k)$ and since $\mathrm{diam}(D_k) \to 0$ as $k \to \infty$ it follows $d(w, w') = 0$ and hence $w = w'$. □

Remark 8.7. Notice that the property of a metric space being complete is *not* preserved by homeomorphism – we have seen that $(0,1)$ is homeomorphic to $\mathbb{R}$ but the former is not complete, while the latter is. This is because a homeomorphism does not have to take Cauchy sequences to Cauchy sequences.

Example 8.8. Let $Y = \{z \in \mathbb{C} : |z| = 1\}\backslash\{1\}$. Then Y is homeomorphic to $(0,1)$ via the map $t \mapsto e^{2\pi i t}$, but their respective closures $\bar{Y}$ and $[0,1]$ however are not homeomorphic. (We will seem a rigorous proof of this later using the notion of connectedness.) The metric spaces Y and $(0,1)$ contain information about their closures in $\mathbb{R}^2$ which is lost when we only consider the topologies the metrics give: the space Y has Cauchy sequences which don't converge in Y, but these all converge to $1 \in \mathbb{C}$, whereas in $(0,1)$ there are two kinds of Cauchy sequences which do not converge in $(0,1)$ – the ones converging to 0 and the ones converging to 1. The point here is that given two Cauchy sequences we can detect if they converge to the same limit without knowing what that the limit actually is: (x_n) and (y_n) converge to the same limit if for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, y_n) < \epsilon$ for all $n \geq N$. Using this idea one can define what is called the *completion* of a metric space (X,d): this is a complete metric space (Y,d) such which X embeds isometrically into as a dense[16] subset. For example, the real numbers $\mathbb{R}$ are the completion of $\mathbb{Q}$.

Many naturally arising metric spaces are complete. We now give a important family of such: recall that if X is any set, the space $\mathscr{B}(X)$ of bounded real-valued functions on X is normed vector space where if $f \in \mathscr{B}(X)$ we define its norm to be $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

Theorem 8.9. *Let X be a set. The normed vector space $(\mathscr{B}(X), \|.\|_\infty)$ is complete.*

Proof. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\mathscr{B}(X)$. Then we have for each $x \in X$

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \to 0,$$

as $n, m \to \infty$. It follows that the sequence $(f_n(x))$ is a Cauchy sequence of real numbers and hence since $\mathbb{R}$ is complete it converges to a real number. Thus we may define a function $f\colon X \to \mathbb{R}$ by setting $f(x) = \lim_{n\to\infty} f_n(x)$.

We claim $f_n \to f$ in $\mathscr{B}(X)$. Note that this requires us to show both that $f \in \mathscr{B}(X)$ and $f_n \to f$ with respect to the norm $\|.\|_\infty$. To check these both hold, fix $\epsilon > 0$. Since (f_n) is Cauchy, we may find an $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n, m \geq N$. Thus we have for all $x \in X$ and $n, m \geq N$

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon.$$

But now letting $n \to \infty$ we see that for any $m \geq N$ we have $|f(x) - f_m(x)| \leq \epsilon$ for all $x \in X$. But then for any such m we certainly have $f - f_m \in \mathscr{B}(X)$ so that[17] $f = f_m + (f - f_m) \in \mathscr{B}(X)$, and since $\|f - f_m\|_\infty \leq \epsilon$ for all $m \geq N$ it follows $f_m \to f$ as $m \to \infty$ as required.

□

As we already observed, if X is also a metric space then we can also consider the space of bounded continuous functions $\mathscr{C}_b(X)$ on X. This is a normed subspace of $\mathscr{B}(X)$, and the following theorem is a generalization of the result you saw last year showing that a uniform limit of continuous functions is continuous (the proof is essentially the same also).

Theorem 8.10. *Let (X,d) be a metric space. The space $\mathscr{C}_b(X)$ is a complete normed vector space.*

Proof. Since we have shown in Theorem 8.9 that $\mathscr{B}(X)$ is complete, by Lemma 8.5 we must show that $\mathscr{C}_b(X)$ is a closed subset of $\mathscr{B}(X)$. Let (f_n) be a Cauchy sequence of bounded continuous functions on X. By Theorem 8.9 this sequence converges to a bounded function $f\colon X \to \mathbb{R}$. We must show that f is continuous. Suppose that $a \in X$ and let $\epsilon > 0$. Then since $f_n \to f$ there is an $N \in \mathbb{N}$ such that $\|f -$

[16] that is, Y is the closure of X.

[17] Recall from Lemma 3.8 that $\mathscr{B}(X)$ is a vector space!

$f_n\|_\infty < \epsilon/3$ for all $n \geq N$. Moreover, if we fix $n \geq N$ then since f_n is continuous, there is a $\delta > 0$ such that $|f_n(x) - f_n(a)| < \epsilon/3$ for all $x \in B(a,\delta)$. But then for $x \in B(a,\delta)$ we have

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

It follows that f is continuous at a, and since a was arbitrary, f is a continuous function as required.

□

Remark 8.11. If X and Y are metric spaces, as we saw in Example 2.12, one can also consider the space $\mathscr{B}(X,Y)$ of bounded functions from X to Y, that is, functions $f\colon X \to Y$ such that $f(X)$ is a bounded subset of Y, along with its subspace $\mathscr{C}_b(X,Y)$ of bounded continuous functions. These are no longer normed vector spaces, but they are both complete metric spaces provided Y is, as you are asked to show in the second problem sheet.

Lemma 8.12. *("Weierstrass M-test"): Let X be a metric space. Suppose that (f_n) is a sequence in $\mathscr{C}_b(X)$ and $(M_n)_{n\geq 0}$ is a sequence of non-negative real numbers such that $\|f_n\|_\infty \leq M_n$ for all $n \in \mathbb{Z}_{\geq 0}$ and $\sum_{n\geq 0} M_n$ exists. Then the series $\sum_{n\geq 0} f_n$ converges in $\mathscr{C}_b(X)$.*

Proof. Let $S_n = \sum_{k=0}^N f_k$ be the sequence of partial sums. Since we know $\mathscr{C}_b(X)$ is complete, it suffices to prove that the sequence $(S_n)_{m\geq 0}$ is Cauchy. But if $n \leq m$ then we have

$$\|S_m - S_n\| \leq \sum_{k=n+1}^{m} \|f_k\| \leq \sum_{k=n+1}^{m} M_k,$$

and since $\sum_{k\geq 0} M_k$ converges, the sum $\sum_{k=n+1}^m M_k$ tends to zero as $m, n \to \infty$ as required. □

Finally, we conclude this section with a theorem which is extremely useful, and is a natural generalization of a result you saw last year in constructive mathematics. We first need some terminology:

Definition 8.13. Let (X,d) and (Y,d) be metric spaces and suppose that $f\colon X \to Y$. We say that f is a *Lipschitz* map (or is *Lipschitz continuous*) if there is a constant $K \geq 0$ such that

$$d(f(x), f(y)) \leq K d(x,y).$$

If $Y = X$ and $K \in [0,1)$ then we say that f is a *contraction mapping* (or simply a *contraction*). Any Lipschitz map is continuous, and in fact uniformly continuous, as is easy to check.

The reason for the restriction of the term contraction to maps from a space to itself is the following theorem. The result is a broad generalization of a result you saw before in the Constructive Mathematics course in Prelims, which you will also see put to good use in the Differential Equations course this term.

Theorem 8.14. *Let (X,d) be a nonempty complete metric space and suppose that $f\colon X \to X$ is a contraction. Then f has a unique fixed point, that is, there is a unique $z \in X$ such that $f(z) = z$.*

Proof. If $y_1, y_2 \in X$ are such that $f(y_1) = y_1$ and $f(y_2) = y_2$ we have $d(y_1,y_2) = d(f(y_1), f(y_2)) \leq K d(y_1,y_2)$ so that $(1-K)d(y_1,y_2) \leq 0$. Since $K \in [0,1)$ and the function d is nonnegative this is possible only if $d(y_1,y_2) = 0$ and hence $y_1 = y_2$. It follows that f has at most one fixed point.

To see that f has a fixed point, fix $a \in X$ and consider the sequence defined by $x_0 = a$ and $x_n = f(x_{n-1})$ for $n \geq 1$. We claim that (x_n) converges and that its limit z is the unique fixed point of f. Indeed if $x_n \to z$ as $n \to \infty$ then since f is continuous we have

$$f(z) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z.$$

Thus z is indeed a fixed point. Thus it remains to show that (x_n) is convergent. Since (X,d) is complete, we need only show that (x_n) is Cauchy. To see this this note first that for $n \geq 1$ we have $d(x_n, x_{n-1}) \leq$

$K^{n-1}d(f(a),a)$ (by induction). But then if $n \geq m$ by the triangle inequality we have

$$d(x_n, x_m) \leq \sum_{k=1}^{n-m} d(x_{m+k}, x_{m+k-1}) \leq d(a, f(a))K^m \sum_{k=1}^{n-m} K^{k-1} \leq \frac{d(a, f(a))}{1-K} K^m,$$

which clearly tends to 0 as $n, m \to \infty$. It follows (x_n) is a Cauchy sequence as required. □

Remark 8.15. This theorem is important not just for the statement, but because the proof shows us how to find the fixed point! (Or rather, at least how to approximate it). The sequence (x_n) in the proof converges to the fixed point, and in fact does so quickly – if we start with an initial guess a, and z is the actual fixed point, then $d(x_n, z) \leq K^n.d(a, z)$.

Remark 8.16. It is worth checking to what extent the hypotheses of the theorem are necessary. One might think of a weaker notion of contraction, for example: if $f\colon X \to X$ has the property that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ then it is easy to see that f has at most one fixed point, but the example $f\colon [1,\infty) \to [1,\infty)$ where $f(x) = x + 1/x$ shows that such a map need not have any fixed points.

The requirement that X is complete is also clearly necessary: if $f\colon (0,1) \to (0,1)$ is given by $f(x) = x/2$ clearly f is a contraction, but f has no fixed points in $(0,1)$.

9. Connected sets

In this section we try to understand what makes a space "connected". There are in fact more than one approaches one can take to this question. We will consider two, and show that for reasonably nice spaces the two notions in fact coincide[18].

The first definition we make tries to capture the fact that the space should not "fall apart" into separate pieces. Since we can always write a space with more than one element as a disjoint union of two subsets, we must take into account the metric, or at least the topology, of our space in making a definition.

Example 9.1. Let $X = [0,1]$ and let $A = [0,1/2)$ and $B = [1/2,1]$. Then clearly $X = A \cup B$ so that X can be divided into two disjoint subsets. However, the point $1/2 \in B$ has points in A arbitrarily close to it, which, intuitively speaking, is why it is "glued" to A.

This suggests that we might say that a decomposition of metric space X into two subsets A and B might legitimately show X to be disconnected if no point of A was a limit point of B and vice versa. This is precisely the content of our definition.

Definition 9.2. Suppose that (X, d) is a metric space. We say that X is *disconnected* if we can write $X = U \cup V$ where U and V are nonempty open subsets of X and $U \cap V = \emptyset$. We say that X is *connected* if it is not disconnected.

Note that if $X = U \cup V$ and U and V are both open and disjoint, then $U = V^c$ is also closed, as is V. Thus U and V also contain all of their limit points, so that no limit point of A lies in B and vice versa.

Remark 9.3. Note that if (X, d) is a metric space and $A \subseteq X$, then the condition that A is connected can be rewritten as follows: if U, V are open in X and $U \cap V \cap A = \emptyset$ then whenever $A \subseteq U \cup V$, either $A \subseteq U$ or $A \subseteq V$.

As the previous remark shows, there are a few ways of expressing the above definition which are all readily seen to be equivalent. We record the most common in the following lemma.

Lemma 9.4. *Let (X, d) be a metric space. The following are equivalent.*

(1) *X is connected.*
(2) *If $f\colon X \to \{0,1\}$ is a continuous function then f is constant.*
(3) *The only subsets of X which are both open and closed are X and $\emptyset$.*

[18] In particular, for the open subsets of the complex plane which are the sets we will be most interested in for second part of the course, the two notions will coincide, but both characterizations of connectedness will be useful.

(Here the set $\{0,1\}$ is viewed as a metric space via its embedding in $\mathbb{R}$, or equivalently with the discrete metric.)

Proof. (1) $\iff$ (2): Let $f\colon X \to \{0,1\}$ be a continuous function. Then since the singleton sets $\{0\}$ and $\{1\}$ are both open in $\{0,1\}$ each of $f^{-1}(0)$ and $f^{-1}(1)$ are open subsets of X which are clearly disjoint. It follows if X is connected then one must be the empty set, and hence f is constant as required. Conversely, if X is not connected then we may write $X = A \cup B$ where A and B are nonempty disjoint open sets. But then the function $f\colon X \to \{0,1\}$ which is 1 on A and 0 on B is non-constant and by the characterization of continuity in terms of open sets, f is clearly continuous.

(1) $\iff$ (3): If X is disconnected then we may write $X = A \cup B$ where A and B are disjoint nonempty open sets. But then $A^c = B$ so that A is closed (as is $B = A^c$) so that A and B proper sets of X which are both open and closed. Conversely, if A is a proper subset of X which is closed and open then A^c is also a proper subset which is both closed and open so that the decomposition $X = A \cup A^c$ shows that X is disconnected. $\square$

Example 9.5. If $X = [0,1] \cup [2,3] \subset \mathbb{R}$ then we have seen that both $[0,1]$ and $[2,3]$ are open in X, hence since X is their disjoint union, X is not connected.

Lemma 9.6. *Let (X,d) be a metric space.*

$i)$ Let $\{A_i : i \in I\}$ be a collection of connected subsets of X such that $\bigcap_{i\in I} A_i \neq \emptyset$. Then $\bigcup_{i\in I} A_i$ is connected.
$ii)$ If $A \subseteq X$ is connected then if B is such that $A \subseteq B \subseteq \bar{A}$, the set B is also connected.
$iii)$ If $f\colon X \to Y$ is continuous and $A \subseteq X$ is connected then $f(A) \subseteq Y$ is connected.

Proof. For the first part, suppose that $f\colon \bigcup_{i\in I} A_i \to \{0,1\}$ is continuous. We must show that f is constant. Pick $x_0 \in \bigcap_{i\in I} A_i$. Then if $x \in \bigcup_{i\in I} A_i$ there is some i for which $x \in A_i$. But then the restriction of f to A_i is constant since A_i is connected, so that $f(x) = f(x_0)$ as $x, x_0 \in A_i$. But since x was arbitrary, it follows that f is constant as required.

See the second problem sheet for hints for the second part.

For the final part, note that since f is continuous, if $f(A) \subseteq U \cup V$ for U and V open in Y with $U \cap V \cap f(A) = \emptyset$, then $A \subset f^{-1}(U) \cup f^{-1}(V)$, $f^{-1}(U) \cap f^{-1}(V) \cap A = \emptyset$ and $f^{-1}(U), f^{-1}(V)$ are open in X. Since A is connected it must lie entirely in one of $f^{-1}(U)$ or $f^{-1}(V)$ and hence $f(A)$ must lie entirely in U or V as required. $\square$

Remark 9.7. Notice that $iii)$ in the previous Lemma implies that if X and Y are homeomorphic, they if X is connected so is Y, and vice versa. Note also that $iii)$ allows us to generalize the characterization of connectedness in terms of functions to the set $\{0,1\}$. We say that a metric (or topological) space is *discrete* if every point is an open set. It is easy to see that the connected subsets of a discrete metric space are precisely the singleton sets, thus any continuous function from a connected set to a discrete set must be constant. This applies for example to sets such as $\mathbb{N}$ and $\mathbb{Z}$, which will be very useful for us later in the course.

Definition 9.8. Part $i)$ of Lemma 9.6 has an important consequence: if (X,d) is a metric space and $x_0 \in X$, then the set of connected subsets of X which contain x_0 is closed under unions, that is, if $\{C_i : i \in I\}$ is any collection of connected subsets containing x_0 then $\bigcup_{i\in I} C_i$ is again a connected subset containing x_0. This means that

$$C_{x_0} = \bigcup_{\substack{C \subseteq X \text{ connected,} \\ x_0 \in C}} C,$$

is the largest[19] connected subset of X which contains x_0, in the sense that any connected subset of X which contains x_0 lies in C_{x_0}. It is called the *connected component* of X containing x_0. The space X is the disjoint union of its connected components.

9.1. **Connected sets in $\mathbb{R}$.**

Proposition 9.9. *The real line $\mathbb{R}$ is connected.*

Proof. Let U and V be open subsets of $\mathbb{R}$ such that $\mathbb{R} = U \cup V$ and $U \cap V = \emptyset$. Suppose for the sake of a contradiction that both U and V are non-empty so that we may pick $x \in U$ and $y \in V$. By symmetry we may assume that $x < y$ (since $U \cap V = \emptyset$ we cannot have $x = y$). Since $[x, y]$ is bounded and $x \in U$, if we let $S = \{z \in [x, y] : z \in U\}$, then $c = \sup(S)$ exists, and certainly $c \in [x, y]$. If $c \in U$ then $c \neq y$ and as U is open there is some $\epsilon_1 > 0$ such that $B(c,\epsilon_1) \subseteq U$. Thus if we set $\delta = \min\{\epsilon_1/2, (y-c)/2\} > 0$ we have $c+\delta \in U \cap [x, y]$ contradicting the fact that c is an upper bound for S. Similarly if $c \in V$ then there is an $\epsilon_2 > 0$ such that $B(c,\epsilon_2) \subseteq V$. But then $\emptyset = (c-\epsilon_2, c] \cap U \supseteq (c-\epsilon_2, c] \cap S$, so that $c-\epsilon_2$ is an upper bound for S, contradiction the fact that c is the least upper bound of S. It follows that one of U or V is the empty set as required. □

Corollary 9.10. *The real line $\mathbb{R}$, every half-line (a,∞), $(-\infty, a)$, $[a,\infty)$ or $(-\infty, a]$ and any interval are all connected subsets of $\mathbb{R}$.*

Proof. We have already seen that $\mathbb{R}$ is connected, and since every open interval (a, b) or open half-line (a,∞), $(-\infty, a)$ is homeomorphic to $\mathbb{R}$ they are also connected. The remaining cases the follow from part ii) of Lemma 9.6. □

Exercise 9.11. Show that any interval or half-line is homeomorphic to one of $[0,1]$, $[0,1)$ or $(0,1)$.

Lemma 9.12. *Suppose that $A \subset \mathbb{R}$ is a connected set. Then A is either $\mathbb{R}$, an interval, or a half-line.*

Proof. Suppose that $x, y \in A$ and $x < y$. We claim that $[x, y] \subseteq A$. Indeed if this is not the case then there is some c with $x < c < y$ and $c \notin A$. But then $A = \big(A \cap (-\infty, c)\big) \cup ((A \cap (c,\infty)\big)$ so that A is not connected.

If we let $\sup(A) = +\infty$ if A is not bounded above and $\inf(A) = -\infty$ if A is not bounded below, then by the approximation property it follows that

$$(\inf(A), \sup(A)) = \bigcup_{\substack{x,y\in A\\ x\leq y}} [x, y] \subseteq A,$$

so that A is an interval or half-line as required. (The $\inf(A)$ and $\sup(A)$ may or may not lie in A, leading to open, closed, or half-open intervals and open or closed half-lines.) □

Proposition 9.13. *(Intermediate Value Theorem.) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the image of f is an interval in $\mathbb{R}$. In particular, f takes every value between $f(a)$ and $f(b)$.*

Proof. Since $[a, b]$ is connected, its image must be connected, and hence by the above it is an interval. The in particular claim follows. □

Remark 9.14. Note that for the Intermediate Value Theorem we only needed to know that $[a, b]$ was connected and that a connected subset A of $\mathbb{R}$ has the property that if $x \leq y$ lie in A then $[x, y] \subseteq A$.

9.2. **Path connectedness.**

A quite different approach to connectedness might start assuming that, whatever a connected set should be, the closed interval should be one[20].

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[19]This is the analogous to the definition of the interior of a subset S of X, which is the largest open subset of X contained in S.

[20]Since we've seen that the closed interval is connected according to our previous definition, it shouldn't be too surprising that we will readily be able to see our second notion of connectedness implies the first. The subtle point will be that it is actually in general a strictly *stronger* condition.

Definition 9.15. Let (X,d) be a metric space. A *path* in X is a continuous function $\gamma\colon [a,b] \to X$ where $[a,b]$ is any non-empty closed interval. If $x, y \in X$ then we say there is a path between x and y if there is a path $\gamma\colon [a,b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. We say that the metric space X is *path-connected* if there is a path between any two points in X. Note that since every closed interval $[a,b]$ is homeomorphic to $[0,1]$ one can equivalently require that paths are continuous functions $\gamma\colon [0,1] \to X$. In the subsequent discussion we will, for convenience, impose this condition.

There are a number of useful operations on paths: Given two paths γ_1, γ_2 in X such that $\gamma_1(1) = \gamma_2(0)$ we can form the *concatenation* $\gamma_1 \star \gamma_2$ of the two paths to be the path

$$\gamma_1 \star \gamma_2(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

Finally, if $\gamma\colon [0,1] \to X$ is a path, then the *opposite* path γ^- is defined by $\gamma^-(t) = \gamma(1-t)$.

Definition 9.16. There is a notion of *path-component* for a metric space: Let us define a relation on points in X as follows: Say $x \sim y$ if there is a path from x to y in X. The constant path $\gamma(t) = x$ (for all $t \in [0,1]$) shows that this relation is reflexive. If γ is a path from x to y then γ^- is a path from y to x, so the relation is symmetric. Finally if γ_1 is a path from x to y and γ_2 is a path from y to z then $\gamma_1 \star \gamma_2$ is a path from x to z, so the relation is transitive. It follows that $\sim$ is an equivalence relation and its equivalence classes, which partition X, are known as the *path components* of X.

We now relate the two notions of connectedness.

Proposition 9.17. *Let (X,d) be a metric space. If X is path-connected then it is connected. If X is an open subset of V where V is a normed vector space, then X is path-connected if it is connected.*

Proof. Suppose that X is path-connected. To see X is connected we use the characterization of connectedness in terms of functions to $\{0,1\}$. Consider such a function $f\colon X \to \{0,1\}$. We wish to show that f is constant, that is, we need to show that if $x, y \in X$ then $f(x) = f(y)$. But Z is path-connected, so there is a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. But then $f \circ \gamma$ is a continuous function from the connected set $[0,1]$ to $\{0,1\}$ so that $f \circ \gamma$ must be constant. But then $f(x) = f \circ \gamma(0) = f \circ \gamma(1) = f(y)$ as required.

Now suppose that X is open in V where V is a normed vector space. Let x_0 be a point in X and let P be its path component. Then if $v \in P$, since X is open, there is an open ball $B(v,r) \subseteq Z$. Given any point w in $B(v,r)$ we have the path $\gamma_w(t) = tw + (1-t)v$ from v to w, and hence concatenating a path from x_0 to v with γ_v we see that w lies in P. It follows that $B(v,r) \subseteq P$ so that P is open in V. But since X is the disjoint union of its path components, it follows that if Z is connected it must have at most one path-component and so is path-connected as required. $\square$

Remark 9.18. Note that it is easy to see that if (X,d) is path-connected and $f\colon X \to Y$ is continuous, then the image of X under f is a path-connected subset of Y: if $y_1 = f(x_1)$ and $y_2 = f(x_2)$ are in the image of f, then if we pick a path $\gamma\colon [0,1] \to X$ from x_1 to x_2 in X, clearly $f \circ \gamma$ is a path from y_1 to y_2 in $f(X)$.

Example 9.19. In general it is not true that a connected set need be path-connected. One reason the two notions differ is because, as well as being connected, the closed interval is what is known as *compact*, a notion we will examine shortly. One consequence of this is that if (X,d) is a metric space and $A \subset X$ is a path-connected subspace then $\bar{A}$, the closure of A need *not* be path-connected, despite the fact that we have already seen that it must be connected.

Consider the subset $A \subseteq \mathbb{R}^2$ given by

$$A = \{(t, \sin(1/t) : t \in (0,1]\}.$$

Since A is clearly the image of $(0,1]$ under a continuous map, it is a connected subset of $\mathbb{R}^2$, and hence its closure $\bar{A} = A \cup (\{0\} \times [-1,1])$ is also connected. We claim however that $\bar{A}$ is *not* path-connected. To see informally why this is the case, suppose $\gamma\colon [0,1] \to \mathbb{R}^2$ has a path from $(1, \sin(1))$ to $(0,1)$. Then the first and second coordinates $x(t)$ and $y(t)$ of γ are continuous functions on a closed interval, so they are uniformly continuous. By the intermediate value theorem $x(t)$ must take every value between 1 and 0, but then $y(t)$ must oscillate between -1 and 1 infinitely often which violates uniform continuity.

10. COMPACTNESS

One of the most fundamental theorems in Prelims Analysis was the Bolzano-Weierstrass theorem on bounded sequences of real numbers. It is the key technical ingredient in a number of the main theorems in the whole sequence – the completeness of the reals, the fact that a continuous function on a closed interval is bounded and attains its bounds, the equivalence of continuity and uniform continuity for functions on a closed interval all rely on it.

In this section we study metric spaces in which the conclusion of the Bolzano-Weierstrass theorem holds, and show that not only do many of the results from Prelims which relied on the Bolanzo-Weierstrass theorem extend to these metric spaces (which is perhaps unsurprising) but also that the class of such spaces is quite rich – it includes for example all closed bounded subsets of $\mathbb{R}^n$ for any n.

Definition 10.1. Let (X,d) be a metric space. We say that X is (*sequentially*[21]) *compact* if any sequence $(x_n)_{n\geq 1}$ in X contains a subsequence $(x_{n_k})_{k\geq 1}$ for which there exists an $\ell \in X$ with $x_{n_k} \to \ell$ as $k \to \infty$.

Example 10.2. You saw last year that any bounded sequence of real numbers contains a convergent subsequence. This readily implies that any closed interval $[a,b] \subset \mathbb{R}$ is compact: Indeed if (x_n) is a sequence in $[a,b]$ then clearly it is bounded, so it contains a convergent subsequence (x_{n_k}), say $x_{n_k} \to \ell$ as $k \to \infty$. But since limits preserve weak inequalities (or in the language we have now developed, $[a,b]$ is a closed subset of $\mathbb{R}$ and so contains its limit points) we must have $\ell \in [a,b]$ and hence $[a,b]$ is compact.

It is also easy to see that $(a,b]$, $[a,b)$ and (a,b) are *not compact* when $b > a$: Take $(a,b]$ for example: a tail of the sequence $(a+1/n)_{n\geq 1}$ will lie in $(a,b]$ and any subsequence of it will converge to $a \notin (a,b]$ since $(a+1/n)_{n\geq 1}$ does, thus $(a+1/n)_{n\geq 1}$ has no subsequence which converges in $(a,b]$.

We now establish some basic properties of compact metric spaces:

Lemma 10.3. *Let (X,d) be a metric space and suppose $Z \subseteq X$ is a subspace.*

(1) *If Z is compact then Z is closed and bounded.*
(2) *If X is compact and Z is closed in X then Z is compact.*

Proof. Suppose that Z is compact in X. If $a \in X$ is a limit point of X then there is a sequence (z_n) in Z which converges to a. Since Z is compact, the sequence (z_n) has a subsequence (z_{n_k}) which converges in Z. But since the limit of a subsequence of a convergent sequence is just the limit of the original sequence we have

$$a = \lim_{n\to\infty} z_n = \lim_{k\to\infty} z_{n_k} \in Z.$$

Thus Z contains all its limit points and hence Z is closed. Next suppose that Z is unbounded in X. Then picking $z_0 \in Z$ we may find $z_n \in Z$ with $d(z_0, z_n) \geq n$ for each $n \in \mathbb{N}$. But then if (z_n) had a convergent subsequence (z_{n_k}) say $z_{n_k} \to b \in Z$ then we would have $d(z_{n_k}, z_0) \geq n_k \geq k$ and also $d(z_{n_k}, z_0) \to d(b, z_0)$, which is a contradiction, since a convergent sequence of real numbers must be bounded.

Now suppose that X is compact and Z is closed in X. Then if (z_n) is a sequence in Z, since X is compact it has a convergent subsequence (z_{n_k}) tending to $c \in X$ say. But then c is a limit point of Z and since Z is closed $c \in Z$, so that (z_n) has a convergent subsequence in Z as required. □

The next Lemma essentially shows that compactness, like connectedness, is a topological property:

[21]The word "compact" is in general used for a notion which is discussed in Section 11. For metric spaces the two notions are equivalent. [Aside: the two notions make sense for arbitrary topological spaces, where they turn out *not* to be equivalent.]

Lemma 10.4. *Let (X, d) and (Y, d) be metric spaces and suppose that $f: X \to Y$ is continuous. Then if X is compact, $f(X)$ is a compact subspace of Y. In particular, if X is compact and $f: X \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

Proof. Suppose that (y_n) is a sequence in $f(X) \subseteq Y$. Then for each n we may pick an $x_n \in X$ such that $f(x_n) = y_n$. Since X is compact the sequence (x_n) contains a convergent subsequence (x_{n_k}) say, with $x_{n_k} \to a$ as $k \to \infty$ for some $a \in X$. But then since f is continuous we have $y_{n_k} = f(x_{n_k}) \to f(a) \in f(X) \subseteq Y$, so that (y_n) has a convergent subsequence whose limit lies in $f(X)$ as required.

For the final sentence, note that $f(X)$ is a compact subset of $\mathbb{R}$ and hence by Lemma 10.3 it is closed and bounded. But this precisely means that the image of f is bounded and attains its bounds as required. □

Remark 10.5. The previous Lemma also shows that compactness is a property which is preserved by homoeomorphisms: If $f: X \to Y$ is a continuous bijection with $g: Y \to X$ its continuous inverse, then if X is compact $f(X) = Y$ must be compact, while conversely if Y is compact then $X = g(Y)$ must be compact.

Theorem 10.6. *Let $f: X \to Y$ be a continuous function and suppose that X is a compact metric space. Then f is uniformly continuous.*

Proof. Suppose for the sake of a contradiction that f is not uniformly continuous. Then there exists some $\epsilon > 0$ such that for each $n \in \mathbb{N}$ we may find $a_n, b_n \in X$ such that $d(a_n, b_n) < 1/n$ but $d(f(a_n), f(b_n)) \geq \epsilon$. Now since X is compact, (a_n) contains a convergent subsequence, (a_{n_k}) say, and since $d(a_{n_k}, b_{n_k}) \leq 1/n_k \leq 1/k$ it follows $\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} b_{n_k} = c$ say. But since f is continuous at c there is a $\delta > 0$ such that for all $x \in X$ with $d(c, x) < \delta$, we have $d(f(c), f(x)) < \epsilon/2$. As both (a_{n_k}) and (b_{n_k}) tend to c, for all sufficiently large k we will have $d(c, a_{n_k}), d(c, b_{n_k}) < \delta$ and hence

$$\epsilon \leq d(f(a_{n_k}), f(b_{n_k})) \leq d(f(a_{n_k}), f(c)) + d(f(c), f(b_{n_k})) < \epsilon/2 + \epsilon/2 < \epsilon,$$

which is a contradiction. Thus f must be uniformly continuous as required. □

10.1. **Compactness and products: a generalization of the Bolzano-Weierstrass theorem.** If (X, d_X) and (Y, d_Y) are metric spaces there are various ways of making their Cartesian product into a metric space. A convenient one for our purposes is the following:

Definition 10.7. Let (X, d_X) and (Y, d_Y) be metric spaces. Define a function d on $(X \times Y)^2$ by setting

$$d((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}.$$

It is immediate that this function satisfies the positivity and symmetry requirements of a metric, and the triangle inequality is also readily checked, so that d gives $X \times Y$ the structure of a metric space.

Example 10.8. Writing $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ this gives us an inductive definition of a metric on $\mathbb{R}^n$. Check that the metric one obtains is the metric d_∞. Since we know this metric is equivalent to the metrics d_1 and d_2 if we can characterize the compact subsets of $\mathbb{R}^n$ equipped with the metric $d = d_\infty$ then we also characterize the compact subsets of $\mathbb{R}^n$ with respect to either d_1 and d_2.

Using the above definition of a metric on products of metric spaces makes the following result easy to check:

Lemma 10.9. *Let X and Y be metric spaces. A sequence $\big((x_n, y_n)\big)_{n\geq 1}$ in $X \times Y$ converges if and only if (x_n) converges in X and (y_n) converges in Y.*

Proof. It is clear from the definitions that the projection maps $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ are continuous (in fact they are Lipschitz continuous with Lipschitz constant 1). It follows that if (x_n, y_n) converges in $X \times Y$ then (x_n) and (y_n) must converge.

Conversely, if $x_n \to a \in X$ and $y_n \to b \in Y$ then

$$d((x_n, y_n), (a, b)) = \max\{d(x_n, a), d(y_n, b)\} \to 0$$

as $n \to \infty$ so that $(x_n, y_n) \to (a, b)$ as $n \to \infty$ as required. □

Proposition 10.10. *Let X and Y be compact metric spaces. Then $X \times Y$ is compact.*

Proof. Let (x_n, y_n) be a sequence in $X \times Y$. As X is compact, the sequence (x_n) in X has a convergent subsequence (x_{n_k}), say $x_{n_k} \to a \in X$ as $k \to \infty$. But then consider the sequence (y_{n_k}) in Y. Since Y is compact this in turn has a convergent subsequence $(y_{n_{k_r}})_{r \geq 1}$, say $y_{n_{k_r}} \to b \in Y$. But since $(x_{n_{k_r}})$ is a subsequence of x_{n_k} is also converges to a and hence by the previous Lemma $(x_{n_{k_r}}, y_{n_{k_r}}) \to (a, b)$ and (x_n, y_n) has a convergent subsequence as required. □

It is now easy to give a generalisation of the Bolzano-Weierstrass theorem to $\mathbb{R}^n$.

Theorem 10.11. *(Bolzano-Weierstrass in $\mathbb{R}^n$). A subset $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. We have already seen in Lemma 10.3 that if X is compact in $\mathbb{R}^n$ then it must be closed and bounded, thus it remains to show that any such set is compact. But if X is bounded then there is an $R > 0$ such that[22] $X \subseteq B(0, R) = [-R, R]^n$. Now by the Bolzano-Weierstrass theorem for $\mathbb{R}$, any closed interval such as $[-R, R]$ is compact. But then using Proposition 10.10 and induction it follows readily that $[-R, R]^n$ is compact, but then again by Lemma 10.3 it follows that X, being a closed subset of a compact metric space, is compact as required. □

Remark 10.12. Note that in a general metric space X, a closed bounded subset of X need *not* be compact. An example of this is given by taking $\mathscr{C}_b(\mathbb{R})$ the normed space of continuous bounded functions on the real line equipped with $\|.\|_\infty$ the supremum metric. If we let

$$f(t) = \begin{cases} 2t, & 0 \leq t \leq 1/2; \\ 2(1-t), & 1/2 \leq t \leq 1 \end{cases}$$

and set $f_n(t) = f(t+n)$ the each f_n is bounded and in fact has $\|f_n\|_\infty = 1$, so that they all lie in $\bar{B}(0, 1)$. However, if $n \neq m$ it is easy to see that $\|f_n - f_m\|_\infty = 1$, so that (f_n) has no convergent subsequence and thus $\bar{B}(0, 1)$ is not compact, despite clearly being closed and bounded in $\mathscr{C}_b(\mathbb{R})$.

[22] Recall that the "open balls" in the d_∞ metric are hypercubes.