Metric spaces and complex analysis

Mathematical Institute, University of Oxford
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Problem Sheet 3

Unless otherwise stated. $\mathbb{R}^n$ *is equipped with the standard topology (and hence notion of convergence and continuity).*

1. Determine which of the following subsets of $\mathbb{R}^2$ are homeomorphic to each other, giving either a homeomorphism in the case where one exists or a proof that one does not exist. (Thus for each pair X_1, X_2 of subsets in the list below, you must decide whether or not X_1 and X_2 are homeomorphic.)

$$\mathbb{R}^2; \quad B(0,1); \quad \bar{B}(0,1); \quad [0,1]\times[0,1]; \quad S^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}; \quad \mathbb{R}\times\{0\}.$$

2. Let $(V, \|.\|)$ be a normed vector space. Show that V has the property that any bounded sequence in V has a convergent subsequence if and only if $S = \{v \in V : \|v\| = 1\}$ is compact. Show moreover that in this case V must be complete. Give an example of a normed vector space which is not complete.

3. Let c_0 denote the normed vector space of bounded sequences equipped with the supremum norm. Show that, given any bounded sequence (x^n) in c_0 there exists a subsequence (x^{n_k}) such that for each coordinate i, the real sequence $(x_i^{n_k})$ converges. Explain why this does *not* show that any such sequence has a convergent subsequence.
[*Hint: Adapt the argument which shows any sequence has a Cauchy subsequence in a totally bounded metric space.*]

4. Let $X = \mathbb{R}$ and suppose that (f_n) is a sequence in $\mathcal{C}_b(X)$ with $\|f_n\|_\infty \leq 1$ for all $n \in \mathbb{N}$. Suppose that for all $\epsilon > 0$ there is a δ such that if $d(x,y) < \delta$ then $d(f_n(x), f_n(y)) < \epsilon$ for all $n \in \mathbb{N}$. Show, by constructing a counter-example, that (f_n) need not have a convergent subsequence.

5. If $a, b \in \mathbb{C}$ are distinct and $k \in (0,1]$ then the locus $L = \{z \in \mathbb{C} : |z-a| = k|z-b|\}$ is a line or circle, depending on whether $k = 1$ or $k < 1$. Show that every circle or line can be realised as such a locus.

6. *i*) Sketch the following subsets of the complex plane:

$$\{z : |z-i| < |z-1|\}; \quad \{z : \mathrm{Im}(\frac{z+i}{2i}) < 0\}; \quad \{z : \mathrm{Re}(z+1) = |z-1|\}; \quad \{e^z : z \in \mathbb{C}\}.$$

ii) Describe geometrically each of the following maps of the complex plane:

$$z \mapsto \bar{z}; \quad z \mapsto e^{i\pi/3}.z; \quad z \mapsto \bar{z} + 2i; \quad z \mapsto iz + 1.$$

The third map is a reflection, say what its invariant line is. The fourth is a rotation, give the angle and centre of rotation.

iii) Which of the following complex sequences converge?

$$(i^n/n); \quad ((-1)^n n/(n+i)); \quad (\frac{n^2+in}{n^2+i}); \quad (e^{ni}).$$

7. Suppose that $f(z) = (az+b)/(cz+d)$ is a Mobius transformation. Find the subgroup Γ of Mob, the group of all Mobius transformations, which consists of those Mobius transformations which are isometries with respect to the standard metric on the Riemann sphere (*i.e.* the one induced from the Euclidean metric on $\mathbb{R}^3$).

[*Hint: In the lecture notes it is shown that the distance on the Riemann sphere corresponds to the distance function on* $\mathbb{P}^1$ *given by*

$$d(L_1, L_2) = 2\sqrt{1 - \frac{|\langle v, w\rangle|^2}{\|v\|^2\|w\|^2|}},$$

where $v \in L_1\backslash\{0\}$ *and* $w \in L_2\backslash\{0\}$. *A Mobius map corresponds to the action of a matrix* $A \in GL_2(\mathbb{C})$ *on lines in* $\mathbb{C}^2$. *Which* 2×2 *matrices automatically preserve the above expression for* d*? Use these to find all of* Γ *in the same way one finds all isometries of* $\mathbb{R}^n$.]