Metric spaces and complex analysis
Mathematical Institute, University of Oxford
Michaelmas Term 2019

Problem Sheet 4

1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function defined by

$$f(x+iy) = \sqrt{|x||y|} \quad \forall x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at $z = 0$ but is not complex differentiable there.

2. Suppose that D is a domain in $\mathbb{C}$ and $f\colon D \to \mathbb{C}$ is a holomorphic[1] function on D. Show that if any one of the conditions

(1) $\Re(f)$ is constant,
(2) $\Im(f)$ is constant,
(3) $|f|$ is constant,

is satisfied, then f is constant.
[*You may assume that a complex differentiable function whose derivative vanishes on D must be constant.*]

3. If $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $S \subset \mathbb{N}$, we say that S is an arithmetic progression if there are integers a, d such that $S = \{a + nd : n \in \mathbb{Z}_{\geq 0}\}$. We call d the *step* of the arithmetic progression. Show that $\mathbb{N}$ cannot be partitioned into finitely many arithmetic progressions with distinct steps (excluding the trivial case of one progression with $a = d = 1$).
[*Hint: Consider the power series $\sum_{n=1}^{\infty} z^n$.*]

4. Suppose that $f(z) = \sum_{n \geq 0} a_n z^n$ is a power series with radius of convergence R. Show that f has a power series expansion about any $z_0 \in B(0, R)$.
[*Hint: Let $z = (z_0 + (z - z_0))$ and use the binomial theorem.*]

5. Consider the multifunction $[F(z)] = (z^2 - 1)^{1/2}$. It was asserted in lectures that $\{1, -1\}$ were branch points for this multifunction. Prove this carefully, using the branches which were constructed in lectures.
[*Hint: Suppose, for the case of -1, there was a continuous branch $f(z)$ of $[F(z)]$ defined on some $B(-1, r)\backslash\{-1\}$. Compare this to a branch as we constructed in lectures to obtain a contradiction.*]

6. *i)* Suppose that $l(z)$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$ and satisfies $\exp l(z) = z$. Show that

$$l(z) = L(z) + 2n\pi i$$

for some $n \in \mathbb{Z}$ where $L(z)$ is the holomorphic branch of log defined in lectures.

ii) Show that there is no holomorphic function $\lambda(z)$ on $\mathbb{C}\backslash\{0\}$ such that $\exp \lambda(z) = z$.

iii) There are unique holomorphic branches of $\log z$, $\sqrt{z}$ and $\sqrt[3]{z}$ on the cut plane $\mathbb{C}\backslash$ {negative imaginary axis} such that $\log 1 = 0$; $\sqrt{1} = 1$; $\sqrt[3]{1} = 1$. For these branches determine

$$\log(1+i), \qquad \sqrt{-1-i}, \qquad \sqrt[3]{-2}, \qquad \sqrt{1-i}.$$

iv) Let C denote the logarithmic spiral given in polar coordinates by $r = 2e^{\theta}$. There is a unique holomorphic branch of log on $\mathbb{C}\backslash C$ such that $\log 1 = 0$. For this branch determine

$$\log i, \qquad \log 3, \qquad \log(-1), \qquad \log 1000, \qquad \log(-1000), \qquad \log 2000.$$

7 (*Optional*). *i)* Suppose that $(a_n)_{n=1}^N$ and $(b_n)_{n=1}^N$ are two finite sequences of complex numbers. Write $B_N = \sum_{n=1}^N b_n$ (taking by convention $B_0 = 0$). Show that for any $1 \leq M \leq N$

$$\sum_{n=M}^{N} a_n b_n = (a_N B_N - a_M B_{M-1}) - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

ii) Now consider the power series $s(z) = \sum_{n=0}^{\infty} \frac{z^n}{n}$. Show that s has radius of convergence 1 and converges on $\{z \in \mathbb{C} : |z| = 1|\}\backslash\{1\}$.

iii) Show that, given any finite subset T of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, there is a power series with radius of convergence 1 which converges on all of $S^1\backslash T$.

[1] *i.e.* f is complex differentiable at each point of D.