14. Complex differentiability and the Cauchy-Riemann equations

We begin by recalling one way of defining the derivative of a real-valued function:

Definition 14.1. Suppose that $E \subseteq \mathbb{R}$ and $f\colon E \to \mathbb{R}$ is a function. If E is a neighbourhood of $x_0 \in \mathbb{R}$ then we say that f is differentiable at x_0 if there is a real number α such that for all $x \in E$ we have

$$f(x) = f(x_0) + \alpha(x - x_0) + \epsilon(x)|x - x_0|,$$

where $\epsilon(x) \to \epsilon(x_0) = 0$ as $x \to x_0$. If α exists it is unique and we write $\alpha = f'(x_0)$.

Remark 14.2. Note that rearranging the above equation we have, for $x \neq a$, $|\epsilon(x)| = |\frac{f(x)-f(a)}{x-a} - \alpha|$, thus the condition that $\epsilon(x) \to 0$ as $x \to a$ is equivalent to $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \alpha$. This also shows the uniqueness of α.

Note also that if E is not a neighbourhood of a, then the above definition still makes sense, but more precise terminology is often used. For example if $E = [a, b]$ with $a < b$ and we take $x_0 = a$ then we say f has a right-hand derivative at x_0 if $\lim_{x\to a}(f(x) - f(a))/(x - a)$ exists as $x \to a$ with $x \in [a, b]$.

The above formulation of the definition of the derivative is a precise formulation of the statement that a function is differentiable at a point a if there is a "best linear approximation", or tangent line, to f near x_0 – that is, the function $x \mapsto f(x_0) + f'(x_0).(x - x_0)$. (The condition that the error term $\epsilon(x)|x - x_0|$ goes to zero faster than x tends to x_0 since $\epsilon(x)$ also tends to zero as x tends to x_0 is the rigorous meaning given to the adjective "best".) This has the advantage that it generalizes immediately to many variables:

Definition 14.3. Suppose that $E \subseteq \mathbb{R}^2$ is an open set, and $f\colon E \to \mathbb{R}^2$. Then we say that f is differentiable at $a \in E$ if there is a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(z) = f(a) + T(z - a) + \epsilon(x)\|z - a\|$$

where $\epsilon(z) \to \epsilon(a) = 0$ as $z \to a$. If such a map T exists it is unique, and we denote it as $Df(a)$ (or sometimes Df_a. It is known as the *total derivative*[29] of f at a.

One can prove the uniqueness of Df_a directly, but it is more illuminating to understand the relation of α to the partial derivatives: If $v \in \mathbb{R}^2$ we define the *directional derivative* of f at a in the direction v to be

$$\partial_v f(a) = \lim_{t\to 0} \frac{f(a + t.v) - f(a)}{t},$$

(if this limit exists). When f is differentiable at a with derivative T, then it follows from the definitions that $t^{-1}(f(a + t.v) - f(a)) = T(v) \pm \epsilon(t.v)\|v\| \to T(v)$ as $t \to 0$, so the directional derivative of f at a all exist. In particular if $z = (x, y)$ and we write $f(z) = (u(x, y), v(x, y)))$ the directional derivatives in the direction of the standard basis vectors e_1 and e_2 are just $(\partial_x u, \partial_x v)$ and $(\partial_y u, \partial_y v)$. Thus we see that if T exists then its matrix with respect to the standard basis is just given by

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

that is the matrix of T is just the *Jacobian matrix* of the partial derivatives of f (and hence the total derivative is uniquely determined, as asserted above).

We are now ready to define what it means for $f\colon U \to \mathbb{C}$ a function on an open subset U of $\mathbb{C}$, to be *complex differentiable*: We simply require that the linear map T is complex linear, or in other words, that T is given by multiplication by a complex number $f'(a)$:

Definition 14.4. A function $f\colon U \to \mathbb{C}$ on an open subset U of $\mathbb{C}$ is complex differentiable at $a \in U$ if there exists a complex number $f'(a)$ such that

$$f(z) = f(a) + f'(a).(z - a) + \epsilon(z).|z - a|,$$

[29] As opposed to the partial derivatives.

where as before $\epsilon(z) \to \epsilon(a) = 0$ as $z \to a$.

Remark 14.5. If a function $f: U \to \mathbb{C}$ on an open subset U of $\mathbb{C}$ is everywhere complex differentiable on U we say it is *holomorphic* on U. We will use the terms "complex differentiable" and "holomorphic" interchangeably. (The term "analytic" is also commonly used, we will come back to that term later.)

Since the standard basis corresponds to $\{1, i\}$, since $(r + is)(x + iy) = (rx - sy) + i(sx + ry)$, the matrix of the linear map given by multiplication by $w = r + is$ is just

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix}$$

This gives us our first important result about complex differentiability:

Lemma 14.6. *(Cauchy-Riemann equations): If U is an open subset of $\mathbb{C}$ and $f: U \to \mathbb{C}$, then f is complex differentiable at $a \in U$ if and only if it is real-differentiable and the partial derivatives satisfy the equations:*

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u.$$

Proof. This follows immediately from the definitions above. Note that it also shows that the complex derivative satisfies $f'(a) = \partial_x f = \partial_x u + i\partial_x v$ and $f'(a) = \frac{1}{i}\partial_y f = \frac{1}{i}(\partial_y u + i\partial_y v)$. □

Remark 14.7. Since the operation of multiplication by a complex number w is a composition of a rotation (by the argument of w) and a dilation (by the modulus of w) the matrix of the corresponding linear map is, up to scalar, a rotation matrix. The Cauchy-Riemann equations just capture this fact for the matrix of the total (real) derivative of a complex differentiable function.

A subtlety of real-differentiability in many variables is that it is possible for the partial derivatives of a function to exist without the function being differentiable in the sense of Definition 14.3. In most reasonable situations however, the following theorem shows that this does not happen:

Theorem 14.8. *Let U be an open subset of $\mathbb{R}^2$ and $f: U \to \mathbb{R}^2$. Let f have components f_1, f_2 so that $f = (f_1, f_2)^t$. If, for $i = 1, 2$, the partial derivatives $\partial_x f_i, \partial_y f_i$ exist and are continuous at $z_0 \in U$ then f is differentiable at z_0.*

The proof of this (although it is not hard – one only needs the definitions and the single-variable mean-value theorem) is not part of this course. For completeness, a proof is given in the Appendix. Combining this theorem with the Cauchy-Riemann equations gives a criterion for complex-differentiability:

Theorem 14.9. *Suppose that U is an open subset of $\mathbb{C}$ and let $f: U \to \mathbb{C}$ be a function. If f is differentiable as a function of two real variables with continuous partial derivatives satisfying the Cauchy-Riemann equations on U, then f is complex differentiable on U.*

Proof. Since the partial derivatives are continuous, Theorem 14.8 shows that f is differentiable as a function of two real variables, with total derivative given by the matrix of partial derivatives. If f also satisfies the Cauchy-Riemann equations, then by Lemma 14.6 it follows it is complex differentiable as required. □

Example 14.10. The previous theorem allows us to show that the complex logarithm is a holomorphic function – up to the issue that we cannot define it continuously on the whole complex plane! The function $z \mapsto e^z$ is not injective, since $e^{z+2n\pi i} = e^z$ for all $n \in \mathbb{Z}$ thus it cannot have an inverse defined on all of $\mathbb{C}$. However, since $e^{x+iy} = e^x(\cos(y) + i\sin(y))$, it follows that if we pick a ray through the origin, say $B = \{z \in \mathbb{C} : \Im(z) = 0, \Re(z) \leq 0\}$, then we may define $\mathrm{Log}: \mathbb{C}\backslash B \to \mathbb{C}$ by setting $\mathrm{Log}(z) = log(|z|) + i\theta$ where $\theta \in (-\pi, \pi]$ is the argument of z. Clearly $e^{\mathrm{Log}(z)} = z$, while $\mathrm{Log}(e^z)$ differs from z by an integer multiple of $2\pi i$.

We claim that Log is complex differentiable: To show this we use Theorem 14.9. Indeed the function $L(x,y) = (\log(\sqrt{x^2+y^2}),\theta) = (L_1, L_2)$ has

$$\partial_x L_1 = \frac{x}{x^2+y^2}, \quad \partial_y L_1 = \frac{y}{x^2+y^2},$$
$$\partial_x L_2 = -\frac{y}{x^2+y^2}, \quad \partial_y L_2 = \frac{x}{x^2+y^2}.$$

where in calculating the partial derivatives of L_2 we used that it is equal to $\arctan(y/x)$ in $(-\pi/2, \pi/2)$ (and one can similarly use other inverse trigonometric functions in the rest of the complex plane). Examining the formulae we see that the partial derivatives are all continuous, and obey the Cauchy-Riemann equations, so that Log is indeed complex differentiable.

14.1. **Harmonic functions.** Recall that the two-dimensional Laplace operator Δ is the differential operator $\partial_x^2 + \partial_y^2$ (defined on functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ which are twice differentiable in the sense that their partial derivatives are again differentiable). A function which is in the kernel of the Laplace operator is said to be *harmonic*, that is, a function $u\colon D \to \mathbb{R}$ defined on an open subset D of $\mathbb{R}^2$ is harmonic if $\Delta(u) = \partial_x^2 u + \partial_y^2 u = 0$.

If we work over the complex numbers, then the Laplacian can be factorized[30] as

$$\Delta = (\partial_x + i\partial_y)(\partial_x - i\partial_y) = (\partial_x - i\partial_y)(\partial_x + i\partial_y).$$

The two first-order differential operators $\partial_x + i\partial_y$ and $\partial_x - i\partial_y$ are closely related to the Cauchy-Riemann equations, as we now show, which yields an important connection between complex-differentiable functions and harmonic functions.

Definition 14.11. The *Wirtinger* (partial) derivatives are defined to be $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. By the equation above, we have $\Delta = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$ (as operators on twice continuously differentiable functions).

Remark 14.12. Notice that, as you study in Differential Equations, to obtain D'Alembert's solution to the one-dimensional wave equation, one factors $\partial_x^2 - \partial_y^2 = (\partial_x - \partial_y)(\partial_x + \partial_y)$, and then performs the change of coordinates $\eta = x + y, \xi = x - y$. Over the complex numbers, the above factorization of Δ shows that we can analyze the Laplacian in a similar way.

Exercise 14.13. Show that if $T\colon \mathbb{C} \to \mathbb{C}$ is any *real linear* map (that is, viewing $\mathbb{C}$ as $\mathbb{R}^2$ we have $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map) then there are unique $a, b \in \mathbb{C}$ such that $T(z) = az + b\bar{z}$. (*Hint: note that the map $z \mapsto az + b\bar{z}$ is $\mathbb{R}$-linear. What matrix does it correspond to as a map from $\mathbb{R}^2$ to itself?*)

Lemma 14.14. *Let U be an open subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$. Then f satisfies the Cauchy-Riemann equations if and only if $\partial_{\bar{z}} f = 0$.*

Proof. Let $f(z) = u(z) + iv(z)$ where u and v are real-valued. Then we have

$$\partial_{\bar{z}} f = (\partial_x + i\partial_y)(u + iv) = (\partial_x u - \partial_y v) + i(\partial_x v + \partial_y u),$$

thus the result follows by taking real and imaginary parts. □

Corollary 14.15. *Suppose that U is an open subset of $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ is complex differentiable and $f(z) = u(z) + iv(z)$ are its real and imaginary parts. If u and v are twice continuously*[31] *differentiable then they are harmonic on U. Moreover any function $g\colon U \to \mathbb{R}$ is harmonic if it is twice continuously differentiable and $\partial_z(g)$ is complex differentiable.*

[30] Acting on functions which are twice continuously differentiable, the two first order factors commute.

[31] That is, all of their second partial deriviatives exist and are continuous.

Proof. The previous Lemma shows that if f is complex differentiable then $\partial_{\bar{z}} f = 0$. Since the Laplacian Δ is equal to $4\partial_z \partial_{\bar{z}}$ it follows that

$$\Delta(\Re(f)) = \Re(\Delta(f)) = \Re(4\partial_z\partial_{\bar{z}}(f)) = 0,$$

so that $\Re(f)$ is harmonic. Similarly we find $\Im(f)$ is harmonic. The final part is also immediate from the fact that $\Delta = 4\partial_{\bar{z}}\partial_z$. $\square$

Remark 14.16. We will shortly see that if $f = u + iv$ is complex differentiable then it is in fact infinitely complex differentiable. Since we have seen that $f' = \partial_x f = \frac{1}{i}\partial_y f$ it follows that u and v are in fact infinitely differentiable so the condition in the previous lemma on the existence and continuity of their second derivatives holds automatically. For a proof of the fact that the mixed partial derivatives of a twice continuously differentiable function are equal, see the Appendix.

Corollary 14.15 motivates the following definition:

Definition 14.17. If $u\colon \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function, we say that $v\colon \mathbb{R}^2 \to \mathbb{R}$ is a *harmonic conjugate* of u if $f(z) = u + iv$ is holomorphic.

Notice that if u is harmonic, it is twice differentiable so that its partial derivatives are continuously differentiable. It follows that a function v is a harmonic conjugate precisely if the pair (u, v) satisfy the Cauchy-Riemann equations. Thus provided we can integrate these equations to find v, a harmonic conjugate will exist. We will show later that, at least when the second partial derivatives are continuous, this can always been done locally in the plane.

14.2. **Power series.** Another important family of examples are the functions which arise from power series. We review here the main results about complex power series which were proved in Analysis II last year:

Definition 14.18. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Then we have an associated sequence of polynomials $s_n(z) = \sum_{k=0}^{n} a_k z^k$. Let S be the set on which this sequence converges pointwise, that is

$$S = \{z \in \mathbb{C} : \lim_{n\to\infty} s_n(z) \text{ exists}\}.$$

Note that since $s_n(0) = a_0$ we have $0 \in S$ so in particular S is nonempty. On the set S, we can define a function $s(z) = \lim_n s_n(z) = \sum_{k=0}^{\infty} a_k z^k$ which we call a *power series*. We define the *radius of convergence* R of the power series $\sum_{k\geq 0} a_k z^k$ to be $\sup\{|z| : z \in S\}$ (or ∞ if S is unbounded).

By convention, given any sequence of complex numbers $(c_n)_{n\geq 0}$ we write $\sum_{k=0}^{\infty} c_k z^k$ for the corresponding power series (even though it may be that it converges only for $z = 0$).

We can give an explicit formula for the radius of convergence using the notion of $\limsup$ which we now recall:

Definition 14.19. If $(a_n)_{n\geq 0}$ is a sequence of real numbers, set $s_n = \sup\{a_k : k \geq n\} \in \mathbb{R} \cup \{\infty\}$ (where we take $s_n = \infty$ if $\{a_k : k \geq n\}$ is not bounded above). Then the sequence (s_n) is either constant and equal to ∞ or eventually becomes a decreasing sequence of real numbers. In the first case we set $\limsup_n a_n = \infty$, whereas in the second case we set $\limsup_n a_n = \lim_n s_n$ (which is finite if (s_n) is bounded below, and equal to $-\infty$ otherwise).

Lemma 14.20. *Let $\sum_{k\geq 0} a_k z^k$ be a power series, let S be the subset of $\mathbb{C}$ on which it converges and let R be its radius of convergence. Then we have*

$$B(0, R) \subseteq S \subseteq \bar{B}(0, R).$$

The series converges absolutely on $B(0, R)$ and if $0 \leq r < R$ then it converges uniformly on $\bar{B}(0, r)$. Moreover, we have

$$1/R = \limsup_n |a_n|^{1/n}.$$

Proof. Let $L = \limsup_n |a_n|^{1/n} \in [0,\infty]$. If $L = 0$ then the statement should be understood to say that the radius of convergence R is ∞, while if $L = \infty$ we take $R = 0$. These two cases are in fact similar but easier than the case where $L \in (0,\infty)$, so we will only give the details for the case where L is finite and positive. Let $s_n = \sup\{|a_k|^{1/k} : k \geq n\}$ so that $L = \lim_{n\to\infty} s_n$.

If $0 < s < 1/L$ we can find an $\epsilon > 0$ such that $(L+\epsilon).s = r < 1$. Thus by definition, for sufficiently large n we have $|a_n|^{1/n} \leq s_n < L+\epsilon$ so that if $|z| \leq s$ we have

$$|a_n||z|^n \leq [(L+\epsilon)|z|]^n \leq r^n,$$

and hence by the comparison test, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly on $\bar{B}(0,s)$. It follows the power series converges everywhere in $B(0,1/L)$.

On the other hand, if $|z| > 1/L$ we can find an $\epsilon_1 > 0$ such that $|z|(L-\epsilon_1) = r > 1$. But then for all k we have $s_k \geq L$ since (s_n) is decreasing, and hence by the approximation property for each k we can find an $n_k \geq k$ with $|a_{n_k}|^{1/n_k} > s_k - \epsilon_1 \geq L - \epsilon$ and hence $|a_{n_k} z^{n_k}| > r^k$. Thus $|a_n z^n|$ has a subsequence which does not tend to zero, so the series cannot converge. It follows the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is $1/L$ as claimed.

□

The next lemma is a relatively straight-forward consequence of standard algebra of limits style results:

Lemma 14.21. *Let $s(z) = \sum_{k=0}^{\infty} a_k z^k$ and $t(z) = \sum_{k=0}^{\infty} b_k z^k$ be power series with radii of convergence R_1 and R_2 respectively and let $T = min\{R_1, R_2\}$.*

(1) *Let $c_n = \sum_{k+l=n} a_k b_l$, then the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence at least T and if $|z| < T$ we have*
$$\sum_{n=0}^{\infty} c_n z^n = s(z)t(z).$$
Thus the product of power series is a power series.

(2) *If $s(z)$ and $t(z)$ are as above, then $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is a power series which converges to $s(z)+t(z)$ in $B(0,T)$, thus the sum of power series is again a power series.*

Proof. This was established in Prelims Analysis II. Note that T is only a lower bound for the radius of convergence in each case – it is easy to find examples where the actual radius of convergence of the sum or product is strictly larger than T. □

The behaviour of a power series at its radius of convergence is in general a rather complicated phenomenon. The following result, which we shall not prove, gives some information however. Some of the ideas involved in its proof are investigated in Problem Set 4.

Theorem 14.22. *(Abel's theorem:) Suppose that (a_n) is a sequence of complex numbers and $\sum_{n=0}^{\infty} a_n$ exists. Then the series $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$ and*

$$\lim_{\substack{r\in(-1,1)\\ r\uparrow 1}} \Big(\sum_{n=0}^{\infty} a_n r^n\Big) = \sum_{n=0}^{\infty} a_n.$$

Proof. Note that since the series $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$ by assumption, its radius of convergence is at least 1, so that the first statement holds. For some idea of what goes into the proof of the second part, see the Problem sets. □

Proposition 14.23. *Let $s(z) = \sum_{k\geq 0} a_k z^k$ be a power series, let S be the domain on which it converges, and let R be its radius of convergence. Then power series $t(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$ also has radius of convergence R and on $B(0,R)$ the power series s is complex differentiable with $s'(z) = t(z)$. In particular, it follows that a power series is infinitely complex differentiable within its radius of convergence.*

Proof. This is proved in Prelims Analysis II. An alternative proof is given in Appendix II. □

Example 14.24. The previous Proposition gives us a large supply of complex differentiable functions. For example,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

are all complex differentiable on the whole complex plane (since $R = \infty$ in each case). Note that one can use the above theorem to show that $\cos(z)^2 + \sin(z)^2 = 1$ for all $z \in \mathbb{C}$, but since $\sin(z)$ and $\cos(z)$ are not in general real, this does not imply that $|\sin(z)|$ or $|\cos(z)|$ at most 1. (In fact it is easy to check that they are both unbounded on $\mathbb{C}$). Using what we have already established about power series it is also easy to check that the complex sin function encompases both the real trigonometric and real hyperbolic functions, indeed:

$$\sin(a + ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b).$$

Example 14.25. Let $s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. Then $s(z)$ has radius of convergence 1, and in $B(0,1)$ we have $s'(z) = \sum_{n=0}^{\infty} z^n = 1/(1-z)$, thus this power series is a complex differentiable function which extends the function $-\log(1-z)$ on the interval $(-1,1)$ to the open disc $B(0,1) \subset \mathbb{C}$. We will see later that we will not be able to extend the function log to a complex differentiable function on $\mathbb{C}\backslash\{0\}$ – we will only be able to construct a "multi-valued" extension.

Note that, slightly more generally, we can work with power series centred at an arbitrary point $z_0 \in \mathbb{C}$. Such power series are functions given by an expression of the form

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

All the results we have shown above immediately extend to these more general power series, since if

$$g(z) = \sum_{n \geq 0} a_n z^n,$$

then the function f is obtained from g simply by composing with the translation $z \mapsto z - z_0$. In particular, the chain rule shows that

$$f'(z) = \sum_{n \geq 1} n a_n (z - z_0)^{n-1}.$$

15. Branch cuts

It is often the case that we study a holomorphic function on a domain $D \subseteq \mathbb{C}$ which does not extend to a function on the whole complex plane.

Example 15.1. Consider the square root "function" $f(z) = z^{1/2}$. Unlike the case of real numbers, every complex number has a square root, but just as for the real numbers, there are two possiblities unless $z = 0$. Indeed if $z = x + iy$ and $w = u + iv$ has $w^2 = z$ we see that

$$u^2 - v^2 = x; \quad 2uv = y,$$

and so

$$u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}, v^2 = \frac{y + \sqrt{x^2 + y^2}}{2}.$$

where the requirement that u^2, v^2 are nonnegative determines the signs. Hence taking square roots we obtain the two possible solutions for w satifying $w^2 = z$. (Note it looks like there are four possible sign combinations in the above, however the requirement that $2uv = y$ means the sign of u determines that of v.) In polars it looks simpler: if $z = re^{i\theta}$ then $w = \pm r^{1/2} e^{i\theta/2}$. Indeed this expression gives us a continuous choice of square root except at the positive real axis: for any $z \in \mathbb{C}$ we may write z uniquely as $re^{i\theta}$ where $\theta \in [0, 2\pi)$, and then set $f(z) = r^{1/2} e^{i\theta/2}$. But now for θ small and positive, $f(z) = r^{1/2} e^{i\theta}$ has small positive argument, but if $z = re^{(2\pi - \epsilon)i}$ we find $f(z) = r^{1/2} e^{(\pi - \epsilon/2)i}$, thus $f(z)$ in the first case is just above the positive real axis, while in the second case $f(z)$ is just below the negative real axis. Thus the function f is only continuous on $\mathbb{C}\backslash\{z \in \mathbb{C} : \Im(z) = 0, \Re(z) > 0\}$. Using Theorem 14.9 you can check f is also holomorphic on this domain. The positive real axis is called a *branch cut* for the *multi-valued function* $z^{1/2}$. By chosing different intervals for the argument (such as $(-\pi, \pi]$ say) we can take different cuts in the plane and obtain different *branches* of the function $z^{1/2}$ defined on their complements.

We formalize these concepts as follows:

Definition 15.2. A *multi-valued function* or *multifunction* on a subset $U \subseteq \mathbb{C}$ is a map $f: U \to \mathscr{P}(\mathbb{C})$ assigning to each point in U a subset[32] of the complex numbers. A *branch* of f on a subset $V \subseteq U$ is a function $g: V \to \mathbb{C}$ such that $g(z) \in f(z)$, for all $z \in V$. If g is continuous (or holomorphic) on V we refer to it as a continuous, (respectively holomorphic) branch of f. We will primarily be interested in branches of multifunctions which are holomorphic.

Remark 15.3. In order to distinguish between multifunctions and functions, it is sometimes useful to introduce some notation: if we wish to consider $z \mapsto z^{1/2}$ as a multifunction, then to emphasize that we mean a multifunction we will write $[z^{1/2}]$. Thus $[z^{1/2}] = \{w \in \mathbb{C} : w^2 = z\}$. Similarly we write $[\mathrm{Log}(z)] = \{w \in \mathbb{C} : e^w = z\}$. This is not a uniform convention in the subject, but is used, for example, in the text of Priestley.

Thus the square root $z \mapsto [z^{1/2}]$ is a multifunction, and we saw above that we can obtain holomorphic branches of it on a cut plane $\mathbb{C}\backslash R$ where $R = \{te^{i\theta} : t \in \mathbb{R}_{\geq 0}\}$. The point here is that both the origin and infinity as "branch points" for the multifunction $[z^{1/2}]$.

Definition 15.4. Suppose that $f: U \to \mathscr{P}(\mathbb{C})$ is a multi-valued function defined on an open subset U of $\mathbb{C}$. We say that $z_0 \in U$ is not a branch point of f if there is an open disk[33] $D \subseteq U$ containing z_0 such that there is a holomorphic branch of f defined on $D\backslash\{z_0\}$. We say z_0 is a *branch point* otherwise. When $\mathbb{C}\backslash U$ is bounded, we say that f does not have a branch point at ∞ if there is a branch of f defined on $\mathbb{C}\backslash B(0,R) \subseteq U$ for some $R > 0$. Otherwise we say that ∞ is a branch point of f.

A *branch cut* for a multifunction f is a curve in the plane on whose complement we can pick a holomorphic branch of f. Thus a branch cut must contain all the branch points.

Example 15.5. Another important example of a multi-valued function which we have already discussed is the complex logarithm: as a multifunction we have $\mathrm{Log}(z) = \{\log(|z|) + i(\theta + 2n\pi) : n \in \mathbb{Z}\}$ where $z = |z|e^{i\theta}$. To obtain a branch of the multifunction we must make a choice of argument function $\arg: \mathbb{C} \to \mathbb{R}$ we may define

$$\mathrm{Log}(z) = \log(|z|) + i\arg(z),$$

which is a continuous function away from the branch cut we chose. By convention, the *principal branch* of Log is defined by taking $\arg(z) \in (-\pi, \pi]$.

Another important class of examples of multifunctions are the *fractional power* multifunctions $z \mapsto [z^\alpha]$ where $\alpha \in \mathbb{C}$: These are given by

$$z \mapsto \exp(\alpha.[\mathrm{Log}(z)]) = \{\exp(\alpha.w) : w \in \mathbb{C}, e^w = z\}$$

Note this is includes the square root multifunction we discussed above, which can be defined without the use of exponential function. Indeed if $\alpha = m/n$ is rational, $m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}$, then $[z^\alpha] = \{w \in \mathbb{C} : w^m = z^n\}$. For $\alpha \in \mathbb{C}\backslash\mathbb{Q}$ however we can only define $[z^\alpha]$ using the exponential function. Clearly from its definition, anytime we choose a branch $L(z)$ of $[Log(z)]$ we obtain a corresponding branch $\exp(\alpha.L(z))$ of $[z^\alpha]$. If $L(z)$ is the principal branch of $[\mathrm{Log}(z)]$ then the corresponding branch of $[z^\alpha]$ is called the *principal branch* of $[z^\alpha]$.

[32]We use the notation $\mathscr{P}(X)$ to denote the *power set* of X, that is, the set of all subsets of X.

[33]In fact any simply-connected domain – see our discussion of the homotopy form of Cauchy's theorem.

Example 15.6. Let $F(z)$ be the multi-function

$$[(1+z)^{\alpha}] = \{\exp(\alpha.w) : w \in \mathbb{C}, \exp(w) = 1+z\}.$$

Using $L(z)$ the principal branch of $[\mathrm{Log}(z)]$ we obtain a branch $f(z)$ of $[(1+z)^{\alpha}]$ given by $f(z) = \exp(\alpha.L(1+z))$. Let $\binom{\alpha}{k} = \frac{1}{k!}\alpha.(\alpha-1)\dots(\alpha-k+1)$. We want to show that a version of the binomial theorem holds for this branch of the multifunction $[(1+z)^{\alpha}]$. Let

$$s(z) = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,$$

By the ratio test, $s(z)$ has radius of convergence equal to 1, so that $s(z)$ defines a holomorphic function in $B(0,1)$. Moreover, you can check using the properties of power series established in the previous section that, within $B(0,1)$, $s(z)$ satisfies $(1+z)s'(z) = a.s(z)$.

Now $f(z)$ is defined on $\mathbb{C}\backslash(-\infty,-1)$, and hence on all of $B(0,1)$. Moreover[34] We claim that within the open ball $B(0,1)$ the power series $s(z) = \sum_{n=0}^{\infty}\binom{\alpha}{k}z^k$ coincides with $f(z)$. Indeed we have

$$\frac{d}{dz}(L(s(z))) = s'(z)/s(z) = \alpha/(1+z) = \frac{d}{dz}(\alpha L(1+z))$$

so that $L(s(z)) = \alpha.L(1+z)+c$ for some constant c (as $B(0,1)$ is connected) which by evaluating at $z=0$ we find is zero. Finally, it follows that $s(z) = \exp(\alpha L(1+z))$ so that $s(z) \in [(1+z)^{\alpha}]$ as required.

Example 15.7. A more interesting example is the function $f(z) = [(z^2-1)^{1/2}]$. Using the principal branch of the square root function, we obtain a branch f_1 of f on the complement of $E = \{z \in \mathbb{C} : z^2-1 \in (-\infty,0]\}$, which one calculates is equal to $(-1,1)\cup i\mathbb{R}$. If we cross either the segment $(-1,1)$ or the imaginary axis, this branch of f is discontinuous.

To find another branch, note that we may write $f(z) = \sqrt{z-1}\sqrt{z+1}$, thus we can take the principal branch of the square root for each of these factors. More explicity, if we write $z = 1+re^{i\theta_1} = -1+se^{i\theta_2}$ where $\theta_1,\theta_2 \in (-\pi,\pi]$ then we get a branch of f given by $f_2(z) = \sqrt{rs}.e^{i(\theta_1+\theta_2)/2}$. Now the factors are discontinuous on $(-\infty,1]$ and $(\infty,-1]$ respectively, however let us examine the behaviour of their product: If z crosses the negative real axis at $\Im(z) < -1$ then θ_1 and θ_2 both jumps by 2π, so that $(\theta_1+\theta_2)/2$ jumps by 2π, and hence $\exp((\theta_1+\theta_2)/2)$ is in fact continuous. On the other hand, if we cross the segment $(-1,1)$ then only the factor $\sqrt{z-1}$ switches sign, so our branch is discontinuous there. Thus our second branch of f is defined away from the cut $[-1,1]$.

Example 15.8. The branch points of the complex logarithm are 0 and infinity: indeed if $z_0 \neq 0$ then we can find a half-plane $H = \{z \in \mathbb{C} : \Im(az) > 0\}$, for some $a \in \mathbb{C}$, $|a| = 1$, such that $z_0 \in H$. We can chose a continuous choice of argument function on H, and this gives a holomorphic branch of Log defined on H and hence on the disk $B(z_0,r)$ for r sufficiently small. The logarithm also has a branch point at infinity, since we cannot chose a continous argument function on $\mathbb{C}\backslash B(0,R)$ for any $R > 0$. (We will return to this point when discussing the winding number later in the course.)

Note that if $f(z) = [\sqrt{z^2-1}]$ then the second of our branches f_2 discussed above shows that f does not have a branch point at infinity, whereas both 1 and -1 are branch points – as we move in a sufficiently small circle around we cannot make a continuous choice of branch. One can given a rigorous proof of this using the branch f_2: given any branch g of $[\sqrt{z^2-1}]$ defined on $B(1,r)$ for $r < 1$ one proves that $g = \pm f_2$ so that g is not continuous on $B(0,r)\cap(-1,1)$. See Problem Sheet 4, question 5, for more details.

Example 15.9. A more sophisticated point of view on branch points and cuts uses the theory of Riemann surfaces. As a first look at this theory, consider the multifunction $f(z) = [\sqrt{z^2-1}]$ again. Let $\Sigma = \{(z,w) \in \mathbb{C}^2 : w^2 = z^2-1\}$ (this is an example of a Riemann surface). Then we have two maps from Σ to $\mathbb{C}$, projecting along the first and second factor: $p_1(z,w) = z$ and $p_2(z,w) = w$. Now if $g(z)$ is a branch of f, it gives us

[34] Any continuous branch $L(z)$ of $[\mathrm{Log}(z)]$ is holomorphic where it is defined and satisfies $\exp(L(z)) = z$, hence by the chain rule one obtains $L'(z) = 1/z$.

a map $G\colon \mathbb{C} \to \Sigma$ where $G(z) = (z, g(z))$. If we take $f_2(z) = \sqrt{z-1}\sqrt{z+1}$ (using the principal branch of the square root function in each case, then let $\Sigma_+\{(z, f_2(z)) : z \notin [-1,1]\}$ and $\Sigma_- = \{(z, -f_2(z)) : z \notin [-1,1]\}$, then $\Sigma_+ \cup \Sigma_-$ covers all of Σ apart from the pairs (z, w) where $z \in [-1,1]$. For such z we have $w = \pm i\sqrt{1-z^2}$, and Σ is obtained by gluing together the two copies Σ_+ and Σ_- of the cut plane $\mathbb{C}\backslash[-1,1]$ along the cut locus $[-1,1]$. However, we must examine the discontinuity of g in order to see how this gluing works: the upper side of the cut in Σ_+ is glued to the lower side of the cut in Σ_- and similarly the lower side of the cut in Σ_+ is glued to the upper side of Σ_-.

Notice that on Σ we have the (single-valued) function $p_2(z, w) = w$, and any map $q\colon U \to \Sigma$ from an open subset U of $\mathbb{C}$ to Σ such that $p_1 \circ q(z) = z$ gives a branch of $f(z) = \sqrt{z^2-1}$ given by $p_2 \circ q$. Such a function is called a *section* of p_1. Thus the multi-valued function on $\mathbb{C}$ becomes a single-valued function on Σ, and a branch of the multifunction corresponds to a section of the map $p_1 : \Sigma \to \mathbb{C}$. In general, given a multi-valued function f one can construct a Riemann surface Σ by gluing together copies of the cut complex plane to obtain a surface on which our multifunction becomes a single-valued function.

16. Paths and Integration

Paths will play a crucial role in our development of the theory of complex differentiable functions. In this section we review the notion of a path and define the integral of a continuous function along a path.

16.1. Paths.

Recall that a *path* in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$. A path is said to be *closed* if $\gamma(a) = \gamma(b)$. If γ is a path, we will write γ^* for its image, that is

$$\gamma^* = \{z \in \mathbb{C} : z = \gamma(t), \text{ some } t \in [a,b]\}.$$

Although for some purposes it suffices to assume that γ is continuous, in order to make sense of the integral along a path we will require our paths to be (at least piecewise) differentiable. We thus need to define what we mean for a path to be differentiable:

Definition 16.1. We will say that a path $\gamma\colon [a,b] \to \mathbb{C}$ is *differentiable* if its real and imaginary parts are differentiable as real-valued functions. Equivalently, γ is differentiable at $t_0 \in [a,b]$ if

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists, and then we denote this limit as $\gamma'(t_0)$. (If $t = a$ or b then we interpret the above as a one-sided limit.) We say that a path is C^1 if it is differentiable and its derivative $\gamma'(t)$ is continuous.

We will say a path is *piecewise* C^1 if it is continuous on $[a,b]$ and the interval $[a,b]$ can be divided into subintervals on each of which γ is C^1. That is, there is a finite sequence $a = a_0 < a_1 < \ldots < a_m = b$ such that $\gamma_{|[a_i,a_{i+1}]}$ is C^1. Thus in particular, the left-hand and right-hand derivatives of γ at a_i $(1 \le i \le m-1)$ may not be equal.

Remark 16.2. Note that a C^1 path may not have a well-defined tangent at every point: if $\gamma\colon [a,b] \to \mathbb{C}$ is a path and $\gamma'(t) \neq 0$, then the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is tangent to γ^*, however if $\gamma'(t) = 0$, the image of γ may have no tangent line there. Indeed consider the example of $\gamma\colon [-1,1] \to \mathbb{C}$ given by

$$\gamma(t) = \begin{cases} t^2 & -1 \le t \le 0 \\ it^2 & 0 \le t \le 1. \end{cases}$$

Since $\gamma'(0) = 0$ the path is C^1, even though it is clear there is no tangent line to the image of γ at 0.

If $s\colon [a,b] \to [c,d]$ is a differentiable map, then we have the following version of the chain rule, which is proved in exactly the same way as the real-valued case. It will be crucial in our definition of the integral of functions $f : \mathbb{C} \to \mathbb{C}$ along paths.

Lemma 16.3. *Let $\gamma\colon [c,d] \to \mathbb{C}$ and $s\colon [a,b] \to [c,d]$ and suppose that s is differentiable at t_0 and γ is differentiable at $s_0 = s(t_0)$. Then $\gamma \circ s$ is differentiable at t_0 with derivative*

$$(\gamma \circ s)'(t_0) = s'(t_0).\gamma'(s(t_0)).$$

Proof. Let $\epsilon\colon [c,d] \to \mathbb{C}$ be given by $\epsilon(s_0) = 0$ and

$$\gamma(x) = \gamma(s_0) + \gamma'(s_0)(x - s_0) + (x - s_0)\epsilon(x),$$

(so that this equation holds for all $x \in [c,d]$), then $\epsilon(x) \to 0$ as $x \to s_0$ by the definition of $\gamma'(s_0)$, *i.e.* ϵ is continuous at t_0. Substituting $x = s(t)$ into this we see that for all $t \neq t_0$ we have

$$\frac{\gamma(s(t)) - \gamma(s_0)}{t - t_0} = \frac{s(t) - s(t_0)}{t - t_0}\big(\gamma'(s(t)) + \epsilon(s(t))\big).$$

Now $s(t)$ is continuous at t_0 since it is differentiable there hence $\epsilon(s(t)) \to 0$ as $t \to t_0$, thus taking the limit as $t \to t_0$ we see that

$$(\gamma \circ s)'(t_0) = s'(t_0)(\gamma'(s_0) + 0) = s'(t_0)\gamma'(s(t_0)),$$

as required. □

Definition 16.4. If $\phi\colon [a,b] \to [c,d]$ is continuously differentiable with $\phi(a) = c$ and $\phi(b) = d$, and $\gamma\colon [c,d] \to \mathbb{C}$ is a C^1-path, then setting $\tilde{\gamma} = \gamma \circ \phi$, by Lemma 16.3 we see that $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ is again a C^1-path with the same image as γ and we say that $\tilde{\gamma}$ is a *reparametrization* of γ.

Definition 16.5. We will say two parametrized paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ are *equivalent* if there is a continuously differentiable bijective function $s\colon [a,b] \to [c,d]$ such that $s'(t) > 0$ for all $t \in [a,b]$ and $\gamma_1 = \gamma_2 \circ s$. It is straight-forward to check that equivalence is indeed an equivalence relation on parametrized paths, and we will call the equivalence classes *oriented curves* in the complex plane. We denote the equivalence class of γ by $[\gamma]$. The condition that $s'(t) > 0$ ensures that the path is traversed in the same direction for each of the parametrizations γ_1 and γ_2. Moreover γ_1 is piecewise C^1 if and only if γ_2 is.

Recall that we saw before (in a general metric space) that any path $\gamma\colon [a,b] \to \mathbb{C}$ has an *opposite* path γ^- and that two paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ can be *concatenated* to give a path $\gamma_1 \star \gamma_2$. If $\gamma, \gamma_1, \gamma_2$ are piecewise C^1 then so are γ^- and $\gamma_1 \star \gamma_2$. (Indeed a piecewise C^1 path is precisely a finite concatenation of C^1 paths).

Remark 16.6. Note that if $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise C^1, then by choosing a reparametrization by a function $\psi\colon [a,b] \to [a,b]$ which is strictly increasing and has vanishing derivative at the points where γ fails to be C^1, we can replace γ by $\tilde{\gamma} = \gamma \circ \psi$ to obtain a C^1 path with the same image. For this reason, some texts insist that C^1 paths have everywhere non-vanishing derivative. In this course we will not insist on this. Indeed sometimes it is convenient to consider a *constant* path, that is a path $\gamma\colon [a,b] \to \mathbb{C}$ such that $\gamma(t) = z_0$ for all $t \in [a,b]$ (and hence $\gamma'(t) = 0$ for all $t \in [a,b]$).

Example 16.7. The most basic example of a closed curve is a circle: If $z_0 \in \mathbb{C}$ and $r > 0$ then the path $z(t) = z_0 + re^{2\pi i t}$ (for $t \in [0,1]$) is the simple closed path with *positive orientation* encircling z_0 with radius r. The path $\tilde{z}(t) = z_0 + re^{-2\pi i t}$ is the simple closed path encircling z_0 with radius r and *negative orientation*.

Another useful path is a line segment: if $a, b \in \mathbb{C}$ then the path $\gamma_{[a,b]}\colon [0,1] \to \mathbb{C}$ given by $t \mapsto a + t(b - a) = (1-t)a + tb$ traverses the line segment from a to b. We denote the corresponding oriented curve by $[a,b]$ (which is consistent with the notation for an interval in the real line). One of the simplest classes of closed paths are triangles: given three points a, b, c, we define the triangle, or triangular path, associated to them, to be the concatenation of the associated line segments, that is $T_{a,b,c} = \gamma_{a,b} \star \gamma_{b,c} \star \gamma_{c,a}$.

16.2. **Integration along a path.** To define the integral of a complex-valued function along a path, we first need to be able to integrate functions $F\colon [a,b] \to \mathbb{C}$ on a closed interval $[a,b]$ taking values in $\mathbb{C}$. Last year in Analysis III the Riemann integral was defined for a function on a closed interval $[a,b]$ taking values in $\mathbb{R}$, but it is easy to extend this to functions taking values in $\mathbb{C}$: Indeed we may write $F(t) = G(t) + iH(t)$ where G, H are functions on $[a,b]$ taking real values. Then we say that F is Riemann integrable if both G and H are, and we define:

$$\int_a^b F(t)dt = \int_a^b G(t)dt + i\int_a^b H(t)dt$$

It is easy to check that the integral is then complex linear, that is, if F_1, F_2 are complex-valued Riemann integrable functions on $[a,b]$, and $\alpha, \beta \in \mathbb{C}$, then $\alpha F_1 + \beta F_2$ is Riemann integrable and

$$\int_a^b (\alpha.F_1 + \beta.F_2)dt = \alpha.\int_a^b F_1 dt + \beta.\int_a^b F_2 dt.$$

Note that if F is continuous, then its real and imaginary parts are also continuous, and so in particular Riemann integrable[35]. The class of Riemann integrable (real or complex valued) functions on a closed interval is however slightly larger than the class of continuous functions, and this will be useful to us at certain points. In particular, we have the following:

Lemma 16.8. *Let $[a,b]$ be a closed interval and $S \subset [a,b]$ a finite set. If f is a bounded continuous function (taking real or complex values) on $[a,b]\backslash S$ then it is Riemann integrable on $[a,b]$.*

Proof. The case of complex-valued functions follows from the real case by taking real and imaginary parts. For the case of a function $f\colon [a,b]\backslash S \to \mathbb{R}$, let $a = x_0 < x_1 < x_2 < \ldots < x_k = b$ be any partition of $[a,b]$ which includes the elements of S. Then on each open interval (x_i, x_{i+1}) the function f is bounded and continuous, and hence integrable. We may therefore set

$$\int_a^b f(t)dt = \int_a^{x_1} f(t)dt + \int_{x_1}^{x_2} f(t)dt + \ldots \int_{x_{k-1}}^{x_k} f(t)dt + \int_{x_k}^{b} f(t)dt.$$

The standard additivity properties of the integral then show that $\int_a^b f(t)dt$ is independent of any choices. $\square$

Remark 16.9. Note that normally when one speaks of a function f being integrable on an interval $[a,b]$ one assumes that f is defined on all of $[a,b]$. However, if we change the value of a Riemann integrable function f at a finite set of points, then the resulting function is still Riemann integrable and its integral is the same. Thus if one prefers the function f in the previous lemma to be defined on all of $[a,b]$ one can define f to take any values at all on the finite set S.

It is easy to check that the Riemann integral of complex-valued functions is complex linear. We also note a version of the triangle inequality for complex-valued functions:

Lemma 16.10. *Suppose that $F\colon [a,b] \to \mathbb{C}$ is a complex-valued function. Then we have*

$$|\int_a^b F(t)dt| \leq \int_a^b |F(t)|dt.$$

Proof. First note that if $F(t) = x(t) + iy(t)$ then $|F(t)| = \sqrt{x^2+y^2}$ so that if F is integrable $|F(t)|$ is also[36]. We may write $\int_a^b F(t)dt = re^{i\theta}$, where $r \in [0,\infty)$ and $\theta \in [0,2\pi)$. Now taking the components of F in the

[35] It is clear this definition extends to give a notion of the integral of a function $f\colon [a,b] \to \mathbb{R}^n$ – we say f is integrable if each of its components is, and then define the integral to be the vector given by the integrals of each component function.

[36] The simplest way to see this is to use that fact that if ϕ is continuous and f is Riemann integrable, then $\phi \circ f$ is Riemann integrable.

direction of $e^{i\theta}$ and $e^{i(\theta+\pi/2)} = ie^{i\theta}$, we may write $F(t) = u(t)e^{i\theta} + iv(t)e^{i\theta}$. Then by our choice of θ we have $\int_a^b F(t)dt = e^{i\theta}\int_a^b u(t)dt$, and so

$$|\int_a^b F(t)dt| = |\int_a^b u(t)dt| \le \int_a^b |u(t)|dt \le \int_a^b |F(t)|dt,$$

where in the first inequality we used the triangle inequality for the Riemann integral of real-valued functions. $\square$