Metric spaces and complex analysis

Mathematical Institute, University of Oxford
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Problem Sheet 5

Throughout this sheet, for $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ we let $\gamma(a,r)$ denote the positively oriented circle centred at a of radius $r > 0$.

1. *Green's Theorem* states, for a region D in the plane, bounded by (an) oriented closed curve(s)[1] C in $\mathbb{R}^2$ and for real-vaued L and M with continuous partial derivatives on D, then

$$\int_C (L\,\mathrm{d}x + M\,\mathrm{d}y) = \int\int_D (M_x - L_y)\,\mathrm{d}x\,\mathrm{d}y.$$

If we assume, for a holomorphic function $f = u+iv$, that u_x, u_y, v_x, v_y are continuous, show that Cauchy's Theorem follows from Green's Theorem, that is, show that for a function f which is holomorphic on the interior D of a closed curve C, we have $\int_C f(z)dz = 0$.
[*The terms "positively oriented" and "interior" should be interpreted as they were in multivariable calculus. We will discuss them more rigorously later in the course.*]

2. By making the substitution $z = re^{i\theta}$, and making clear any special cases, for each integer k determine $\int_{\gamma(0,r)} z^k\,\mathrm{d}z$ (where as usual $\gamma(0,r)$ is the path $\gamma(0,r)(t) = re^{it}$ for $t \in [0, 2\pi]$). By writing $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/2i$ rewrite the integral on the left as a path integral around $\gamma(0,1)$ and deduce that

$$\int_0^{2\pi} \sin^{2n}\theta\,\mathrm{d}\theta = \frac{2\pi}{4^n}\binom{2n}{n}.$$

3. Use the estimation lemma to show that, if $\gamma\colon [0,1] \to \mathbb{C}$ is a closed path, then the winding number $I(\gamma, z)$ is constant on the connected components of $\mathbb{C}\backslash\gamma^*$.
[*Hint: Since it is integer-valued, it suffices to show that $I(\gamma,z)$ is a continuous function on $\mathbb{C}^\times\backslash\gamma^*$.*]

4. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic on $U\backslash\{p\}$ for some $p \in U$, and that f is bounded near p (thus there are constants $r, K \in \mathbb{R}_{>0}$ such that $|f(z)| < K$ for all $z \in B(p,r)$). Show that if T is any triangle whose interior is entirely contained in U then Cauchy's theorem for a triangle still holds, that is

$$\int_T f(z)dz = 0.$$

[*Hint: Use Cauchy's theorem for a triangle to shrink the size of the triangle T.*]

5. Use Cauchy's Integral Formula and the holomorphic function $f(z) = z^n(z-a)^{-1}$, where $a \in \mathbb{R}$ with $a > 1$, to calculate the integral:

$$\int_0^{2\pi} \frac{\cos(n\theta)}{1 - 2a\cos(\theta) + a^2}d\theta.$$

6. Suppose that D is a domain bounded by a contour C, which we assume can be parameterized by a function $\gamma_1\colon [0,1] \to \mathbb{C}$ (that is, $C = \gamma_1^*$). Let $z_0 \in D$ and let $r > 0$ be small enough so that $\bar{B}(z_0, r) \subset D$. The region $D\backslash\bar{B}(z_0,r)$ is thus bounded by $C \cup \partial B(z_0,r)$. Use the result of question 1 to show that if f is holomorphic on $D\backslash\{z_0\}$ then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

where $\gamma_2(t) = z_0 + re^{it}$, $(0 \le t \le 2\pi)$.

Use this and question 2 to calculate

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}.$$

7. In this question you should compute winding numbers "by eye".
   (1) Compute the value of the winding number of the path γ in the connected components of $\mathbb{C}\backslash\gamma^*$ in the diagram below.
   (2) Suppose that $a, b \in \mathbb{C}$ and $|a| < r < |b|$. Compute using the integral formula for the winding number, the integral

$$\int_{\gamma(0,r)} \frac{dz}{(z-a)(z-b)}.$$

[1] Note that the boundary of a region in the plane, for example "with holes", may be a disjoint union of closed curves.

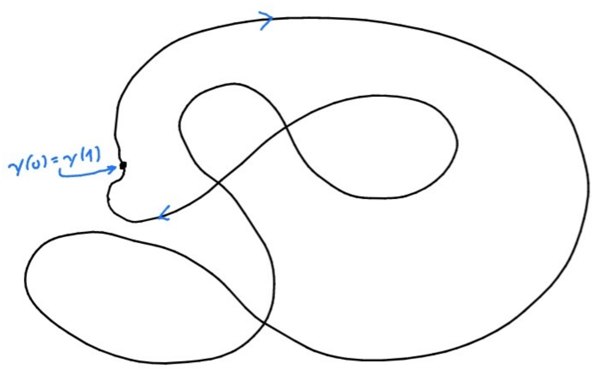

FIGURE 1. The path γ for question 7.