We are now ready to define the integral of a function $f\colon \mathbb{C} \to \mathbb{C}$ along a piecewise-C^1 curve.

Definition 16.11. If $\gamma\colon [a,b] \to \mathbb{C}$ is a piecewise-C^1 path and $f\colon \mathbb{C} \to \mathbb{C}$, then we define the integral of f along γ to be

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

In order for this integral to exist in the sense we have defined, we have seen that it suffices for the functions $f(\gamma(t))$ and $\gamma'(t)$ to be bounded and continuous at all but finitely many t. Our definition of a piecewise C^1-path ensures that $\gamma'(t)$ is bounded and continuous away from finitely many points (the boundedness follows from the existence of the left and right hand limits at points of discontinuity of $\gamma'(t)$). For most of our applications, the function f will be continuous on the whole image γ^* of γ, but it will occasionally be useful to weaken this to allow $f(\gamma(t))$ finitely many (bounded) discontinuities.

Lemma 16.12. *If $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 path and $\tilde{\gamma}\colon [c,d] \to \mathbb{C}$ is an equivalent path, then for any continuous function $f\colon \mathbb{C} \to \mathbb{C}$ we have*

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

In particular, the integral only depends on the oriented curve $[\gamma]$.

Proof. Since $\tilde{\gamma}$ is equivalent to γ there is a continuously differentiable function $s\colon [c,d] \to [a,b]$ with $s(c) = a$, $s(d) = b$ and $s'(t) > 0$ for all $t \in [c,d]$. Suppose first that γ is C^1. Then by the chain rule we have

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\gamma(s(t)))(\gamma\circ s)'(t)dt \\ &= \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \int_a^b f(\gamma(s))\gamma'(s)ds \\ &= \int_\gamma f(z)dz.\end{aligned}$$

where in the second last equality we used the change of variables formula. If $a = x_0 < x_1 < \ldots < x_n = b$ is a decomposition of $[a,b]$ into subintervals such that γ is C^1 on $[x_i, x_{i+1}]$ for $1 \leq i \leq n-1$ then since s is a continuous increasing bijection, we have a corresponding decomposition of $[c,d]$ given by the points $s^{-1}(x_0) < \ldots < s^{-1}(x_n)$, and we have

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \sum_{i=0}^{n-1}\int_{s^{-1}(x_i)}^{s^{-1}(x_{i+1})} f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(\gamma(x))\gamma'(x)dx \\ &= \int_a^b f(\gamma(x))\gamma'(x)dx = \int_\gamma f(z)dz.\end{aligned}$$

where the third equality follows from the case of C^1 paths established above. $\square$

Definition 16.13. If $\gamma\colon [a,b] \to \mathbb{C}$ is a C^1 path then we define the *length* of γ to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Using the chain rule as we did to show that the integrals of a function $f: \mathbb{C} \to \mathbb{C}$ along equivalent paths are equal, one can check that the length of a parametrized path is also constant on equivalence classes of paths, so in fact the above defines a length function for oriented curves. The definition extends in the obvious way to give a notion of length for piecewise C^1-paths. More generally, one can define the integral *with respect to arc-length* of a function $f: U \to \mathbb{C}$ such that $\gamma^* \subseteq U$ to be

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt.$$

This integral is invariant with respect to C^1 reparametrizations $s: [c,d] \to [a,b]$ if we require $s'(t) \neq 0$ for all $t \in [c,d]$ (the condition $s'(t) > 0$ is not necessary because of this integral takes the modulus of $\gamma'(t)$). In particular $\ell(\gamma) = \ell(\gamma^-)$.

The integration of functions along piecewise smooth paths has many of the properties that the integral of real-valued functions along an interval possess. We record some of the most standard of these:

Proposition 16.14. *Let $f, g: U \to \mathbb{C}$ be continuous functions on an open subset $U \subseteq \mathbb{C}$ and $\gamma, \eta: [a,b] \to \mathbb{C}$ be piecewise-C^1 paths whose images lie in U. Then we have the following:*

(1) *(Linearity): For $\alpha, \beta \in \mathbb{C}$,*

$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$

(2) *If γ^- denotes the opposite path to γ then*

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

(3) *(Additivity): If $\gamma \star \eta$ is the concatenation of the paths γ, η in U, we have*

$$\int_{\gamma\star\eta} f(z)dz = \int_\gamma f(z)dz + \int_\eta f(z)dz.$$

(4) *(Estimation Lemma.) We have*

$$|\int_\gamma f(z)dz| \leq \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).$$

Proof. Since f, g are continous, and γ, η are piecewise C^1, all the integrals in the statement are well-defined: the functions $f(\gamma(t))\gamma'(t)$, $f(\eta(t))\eta'(t)$, $g(\gamma(t))\gamma'(t)$ and $g(\eta(t))\eta'(t)$ are all Riemann integrable. It is easy to see that one can reduce these claims to the case where γ is smooth. The first claim is immediate from the linearity of the Riemann integral, while the second claim follows from the definitions and the fact that $(\gamma^-)'(t) = -\gamma'(a+b-t)$. The third follows immediately for the corresponding additivity property of Riemann integrable functions.

For the fourth part, first note that $\gamma([a,b])$ is compact in $\mathbb{C}$ since it is the image of the compact set $[a,b]$ under a continuous map. It follows that the function $|f|$ is bounded on this set so that $\sup_{z\in\gamma([a,b])} |f(z)|$ exists. Thus we have

$$\begin{aligned}
|\int_\gamma f(z)dz| &= |\int_a^b f(\gamma(t))\gamma'(t)dt| \\
&\leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\
&\leq \sup_{z\in\gamma^*} |f(z)| \int_a^b |\gamma'(t)|dt \\
&= \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).
\end{aligned}$$

where for the first inequality we use the triangle inequality for complex-valued functions as in Lemma 16.10 and the positivity of the Riemann integral for the second inequality. □

Remark 16.15. We give part (4) of the above proposition a name (the "estimation lemma") because it will be very useful later in the course. We will give one important application of it now:

Proposition 16.16. *Let $f_n\colon U \to \mathbb{C}$ be a sequence of continuous functions on an open subset U of the complex plane. Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a path whose image is contained in U. If (f_n) converges uniformly to a function f on the image of γ then*

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz.$$

Proof. We have

$$\begin{aligned}\left|\int_\gamma f(z)dz - \int_\gamma f_n(z)dz\right| &= \left|\int_\gamma (f(z) - f_n(z))dz\right| \\ &\leq \sup_{z\in\gamma^*}\{|f(z) - f_n(z)|\}.\ell(\gamma),\end{aligned}$$

by the estimation lemma. Since we are assuming that f_n tends to f uniformly on γ^* we have $\sup\{|f(z) - f_n(z)| : z \in \gamma^*\} \to 0$ as $n \to \infty$ which implies the result. □

Definition 16.17. Let $U \subseteq \mathbb{C}$ be an open set and let $f\colon U \to \mathbb{C}$ be a continuous function. If there exists a differentiable function $F\colon U \to \mathbb{C}$ with $F'(z) = f(z)$ then we say F is a *primitive* for f on U.

The fundamental theorem of calculus has the following important consequence[37]:

Theorem 16.18. *(Fundamental theorem of Calculus): Let $U \subseteq \mathbb{C}$ be a open and let $f\colon U \to \mathbb{C}$ be a continuous function. If $F\colon U \to \mathbb{C}$ is a primitive for f and $\gamma\colon [a,b] \to U$ is a piecewise C^1 path in U then we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular the integral of such a function f around any closed path is zero.

Proof. First suppose that γ is C^1. Then we have

$$\begin{aligned}\int_\gamma f(z)dz = \int_\gamma F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \frac{d}{dt}(F\circ\gamma)(t)dt \\ &= F(\gamma(b)) - F(\gamma(a)),\end{aligned}$$

where in second line we used a version of the chain rule[38] and in the last line we used the Fundamental theorem of Calculus from Prelims analysis on the real and imaginary parts of $F \circ \gamma$.

[37] You should compare this to the existence of a potential in vector calculus.

[38] See the appendix for a discussion of this – we need a version of the chain rule for a composition of real-differentiable functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R} \to \mathbb{R}^2$.

If γ is only[39] piecewise C^1, then take a partition $a = a_0 < a_1 < \ldots < a_k = b$ such that γ is C^1 on $[a_i, a_{i+1}]$ for each $i \in \{0, 1, \ldots, k-1\}$. Then we obtain a telescoping sum:

$$\begin{aligned}\int_\gamma f(z) &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt \\ &= \sum_{i=0}^{k-1} (F(\gamma(a_{i+1})) - F(\gamma(a_i))) \\ &= F(\gamma(b)) - F(\gamma(a)),\end{aligned}$$

Finally, since γ is closed precisely when $\gamma(a) = \gamma(b)$ it follows immediately that the integral of f along a closed path is zero. □

Remark 16.19. If $f(z)$ has finitely many point of discontinuity $S \subset U$ but is bounded near them, and $\gamma(t) \in S$ for only finitely many t, then provided F is continuous and $F' = f$ on $U \backslash S$, the same proof shows that the fundamental theorem still holds – one just needs to take a partition of $[a, b]$ to take account of those singularities along with the singularities of $\gamma'(t)$.

Theorem 16.18 already has an important consequence:

Corollary 16.20. *Let U be a domain and let $f\colon U \to \mathbb{C}$ be a function with $f'(z) = 0$ for all $z \in U$. Then f is constant.*

Proof. Pick $z_0 \in U$. Since U is path-connected, if $w \in U$, we may find[40] a piecewise C^1-path $\gamma\colon [0,1] \to U$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$. Then by Theorem 16.18 we see that

$$f(w) - f(z_0) = \int_\gamma f'(z)dz = 0,$$

so that f is constant as required. □

The following theorem is a kind of converse to the fundamental theorem:

Theorem 16.21. *If U is a domain (i.e. it is open and path connected) and $f\colon U \to \mathbb{C}$ is a continuous function such that for any closed path in U we have $\int_\gamma f(z)dz = 0$, then f has a primitive.*

Proof. Fix z_0 in U, and for any $z \in U$ set

$$F(z) = \int_\gamma f(z)dz.$$

where $\gamma\colon [a, b] \to U$ with $\gamma(a) = z_0$ and $\gamma(b) = z$.

We claim that $F(z)$ is independent of the choice of γ. Indeed if γ_1, γ_2 are two such paths, let $\gamma = \gamma_1 \star \gamma_2^-$ be the path obtained by concatenating γ_1 and the opposite γ_2^- of γ_2 (that is, γ traverses the path γ_1 and then goes backward along γ_2). Then γ is a closed path and so, using Proposition 16.14 we have

$$0 = \int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2^-} f(z)dz,$$

hence since $\int_{\gamma_2^-} f(z)dz = -\int_{\gamma_2} f(z)dz$ we see that $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

Next we claim that F is differentiable with $F'(z) = f(z)$. To see this, fix $w \in U$ and $\epsilon > 0$ such that $B(w,\epsilon) \subseteq U$ and choose a path $\gamma\colon [a, b] \to U$ from z_0 to w. If $z_1 \in B(w,\epsilon) \subseteq U$, then the concatenation of γ

[39] The reason we must be careful about this case is that the Fundamental Theorem of Calculus only holds when the integrand is continuous.

[40] Check that you see that if U is an open subset of $\mathbb{C}$ which is path-connected then any two points can be joined by a piecewise C^1-path.

with the straight-line path $s\colon [0,1] \to U$ given by $s(t) = w + t(z-w)$from w to z is a path γ_1 from z_0 to z. It follows that

$$\begin{aligned}
F(z_1) - F(w) &= \int_{\gamma_1} f(z)dz - \int_\gamma f(z)dz \\
&= (\int_\gamma f(z)dz + \int_s f(z)dz) - \int_\gamma f(z)dz \\
&= \int_s f(z)dz.
\end{aligned}$$

But then we have for $z_1 \neq w$

$$\begin{aligned}
\left|\frac{F(z_1)-F(w)}{z_1-w} - f(w)\right| &= \left|\frac{1}{z_1-w}\left(\int_0^1 f(w+t(z_1-w)(z_1-w)dt\right) - f(w)\right| \\
&= \left|\int_0^1 (f(w+t(z_1-w)) - f(w))dt\right| \\
&\leq \sup_{t\in[0,1]} |f(w+t(z_1-w)) - f(w)| \\
&\to 0 \text{ as } z_1 \to w
\end{aligned}$$

as f is continuous at w. Thus F is differentiable at w with derivative $F'(w) = f(w)$ as claimed. □

Remark 16.22. Note that any two primitives for a function f differ by a constant: This follows immediately from Corollary 16.20, since if F_1 and F_2 are two primitives, their difference $(F_1 - F_2)$ has zero derivative.

17. Winding numbers

The previous section on the fundamental theorem of calculus in the complex plane shows that not every holomorphic function can have a primitive. The most fundamental example of this is the function $f(z) = 1/z$ on the domain $\mathbb{C}^\times$.

Example 17.1. Let $f\colon \mathbb{C}^\times \to \mathbb{C}^\times$ be the function $f(z) = 1/z$. Then f does not have a primitive on $\mathbb{C}^\times$. Indeed if $\gamma\colon [0,1] \to \mathbb{C}$ is the path $\gamma(t) = \exp(2\pi i t)$ then

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 \frac{1}{\exp(2\pi i t)}.(2\pi i \exp(2\pi i t))dt = 2\pi i.$$

Since the path γ is closed, this integral would have to be zero if $f(z)$ has a primitive in an open set containing γ^*, thus $f(z)$ has no primitive on $\mathbb{C}^\times$ as claimed.

Note that $1/z$ *does* have a primitive on any domain in $\mathbb{C}^\times$ where we can chose a branch of the argument function (or equivalently a branch of [Log(z)]): Indeed if $l(z)$ is a branch of [Log(z)] on a domain $D \subset \mathbb{C}^\times$ then since $\exp(l(z)) = z$ the chain rule shows that $\exp(l(z)).l'(z) = 1$ and hence $l'(z) = 1/z$.

In the present section we investigate the change in argument as we move along a path. It will turn out to be a basic ingredient in computing integrals around closed paths. In more detail, suppose that $\gamma\colon [0,1] \to \mathbb{C}$ is a closed path which does not pass through 0. We would like to give a rigorous definition of the number of times γ "goes around the origin". Roughly speaking, this will be the change in argument $\arg(\gamma(t))$, and therein lies the difficulty, since $\arg(z)$ cannot be defined continuously on all of $\mathbb{C}\backslash\{0\}$. The next Proposition shows that we *can* however always define the argument as a continuous function *of the parameter* $t \in [0,1]$:

Proposition 17.2. *Let $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ be a path. Then there is continuous function $a\colon [0,1] \to \mathbb{R}$ such that*

$$\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}.$$

Moreover, if a and b are two such functions, then there exists $n \in \mathbb{Z}$ such that $a(t) = b(t) + n$ for all $t \in [0,1]$.

Proof. By replacing $\gamma(t)$ with $\gamma(t)/|\gamma(t)|$ we may assume that $|\gamma(t)| = 1$ for all t. Since γ is continuous on a compact set, it is uniformly continuous, so that there is a $\delta > 0$ such that $|\gamma(s)-\gamma(t)| < \sqrt{3}$ for any s, t with $|s-t| < \delta$. Choose an integer $n > 0$ such that $n > 1/\delta$ so that on each subinterval $[i/n, (i+1)/n]$ we have $|\gamma(s)-\gamma(t)| < \sqrt{3}/2$. Now on any half-plane in $\mathbb{C}$ we may certainly define a holomorphic branch of $[\mathrm{Log}(z)]$ (simply pick a branch cut along a ray in the opposite half-plane) and hence a continuous argument function, and if $|z_1| = |z_2| = 1$ and $|z_1 - z_2| < \sqrt{3}$, then the angle between z_1 and z_2 is at most $\pi/3$. It follows there exists a continuous functions $a_i \colon [j/n, (j+1)/n] \to \mathbb{R}$ such that $\gamma(t) = e^{2\pi i a_j(t)}$ for $t \in [j/n, (j+1)/n]$ (since $\gamma([j/n, (j+1)/n])$ must lie in an arc of length at most $2\pi/3$). Now since $e^{2\pi i a_j(j/n)} = e^{2\pi i a_{j-1}(j/n)}$ $a_{j-1}(j/n)$ and $a_i(j/n)$ differ by an integer. Thus we can successively adjust the a_j for $j > 1$ by an integer (as if $\gamma(t) = e^{2\pi i a_j(t)}$ then $\gamma(t) = e^{2\pi i (a(t)+n)}$ for any $n \in \mathbb{Z}$) to obtain a continuous function $a \colon [0,1] \to \mathbb{C}$ such that $\gamma(t) = e^{2\pi i a(t)}$ as required. Finally, the uniqueness statement follows because $e^{2\pi i (a(t)-b(t))} = 1$, hence $a(t) - b(t) \in \mathbb{Z}$, and since $[0,1]$ is connected it follows $a(t) - b(t)$ is constant as required. □

Definition 17.3. If $\gamma \colon [0,1] \to \mathbb{C}\backslash\{0\}$ is a closed path and $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}$ as in the previous lemma, then since $\gamma(0) = \gamma(1)$ we must have $a(1) - a(0) \in \mathbb{Z}$. This integer is called the *winding number* $I(\gamma, 0)$ of γ around 0. It is uniquely determined by the path γ because the function a is unique up to an integer. By translation, if γ is any closed path and z_0 is not in the image of γ, we may define the winding number $I(\gamma, z_0)$ of γ about z_0 in the same fashion. Explicitly, if γ is a closed path with $z_0 \notin \gamma^*$ then let $t \colon \mathbb{C} \to \mathbb{C}$ be given by $t(z) = z - z_0$ and define $I(\gamma, z_0) = I(t \circ \gamma, 0)$.

Remark 17.4. Note that if $\gamma \colon [0,1] \to U$ where $0 \notin U$ and there exists a holomorphic branch $L \colon U \to \mathbb{C}$ of $[\mathrm{Log}(z)]$ on U, then $I(\gamma, 0) = 0$. Indeed in this case we may define $a(t) = \Im(L(\gamma(t)))$, and since $\gamma(0) = \gamma(1)$ it follows $a(1) - a(0) = 0$ as claimed. Note also that the definition of the winding number only requires the closed path γ to be continuous, not piecewise C^1. Of course as usual, we will mostly only be interested in piecewise C^1 paths, as these are the ones along which we can integrate functions.

We now see that the winding number has a natural interpretation in term of path integrals: Note that if γ is piecewise C^1 then the function $a(t)$ is also piecewise C^1, since any branch of the logarithm function is in fact differentiable where it is defined, and $a(t)$ is locally given as $\Im(\log(\gamma(t))$ for a suitable branch.

Lemma 17.5. *Let γ be a piecewise C^1 closed path and $z_0 \in \mathbb{C}$ a point not in the image of γ. Then the winding number $I(\gamma, z_0)$ of γ around z_0 is given by*

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Proof. If $\gamma \colon [0,1] \to \mathbb{C}$ we may write $\gamma(t) = z_0 + r(t)e^{2\pi i a(t)}$ (where $r(t) = |\gamma(t) - z_0| > 0$ is continuous and the existence of $a(t)$ is guaranteed by Proposition 17.2). Then we have

$$\begin{aligned}
\int_\gamma \frac{dz}{z - z_0} &= \int_0^1 \frac{1}{r(t)e^{2\pi i a(t)}} . \left(r'(t) + 2\pi i r(t) a'(t)\right) e^{2\pi i a(t)} dt \\
&= \int_0^1 r'(t)/r(t) + 2\pi i a'(t) dt = [\log(r(t)) + 2\pi i a(t)]_0^1 \\
&= 2\pi i (a(1) - a(0)),
\end{aligned}$$

since $r(1) = r(0) = |\gamma(0) - z_0|$. □

The next Proposition will be useful not only for the study of winding numbers. We first need a definition:

Definition 17.6. If $f \colon U \to \mathbb{C}$ is a function on an open subset U of $\mathbb{C}$, then we say that f is *analytic* on U if for every $z_0 \in \mathbb{C}$ there is an $r > 0$ with $B(z_0, r) \subseteq U$ such that there is a power series $\sum_{k=0}^\infty a_k (z - z_0)^k$ with radius of convergence at least r and $f(z) = \sum_{k=0}^\infty a_k (z - z_0)^k$. An analytic function is holomorphic, as any power series is (infinitely) complex differentiable.

Proposition 17.7. *Let U be an open set in $\mathbb{C}$ and let $\gamma\colon [0,1] \to U$ be a closed path. If $f(z)$ is a continuous function on γ^* then the function*

$$I_f(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz,$$

is analytic. in w.

In particular, if $f(z) = 1$ this shows that the function $w \mapsto I(\gamma, w)$ is a continuous function on $\mathbb{C}\backslash\gamma^*$, and hence, since it is integer-valued, it is constant on the connected components of $\mathbb{C}\backslash\gamma^*$.

Proof. We wish to show that $I_\gamma(f(w)$ is holomorphic at each $z_0 \in \mathbb{C}\backslash\gamma^*$. Translating if necessary we may assume $z_0 = 0$.

Now since $\mathbb{C}\backslash\gamma^*$ is open, there is some $r > 0$ such that $B(0,2r) \cap \gamma^* = \emptyset$. We claim that $I_f(\gamma, w)$ is holomorphic in $B(0.r)$. Indeed if $w \in B(0,r)$ and $z \in \gamma^*$ it follows that $|w/z| < 1/2$. Moreover, since γ^* is compact, $M = \sup\{|f(z)| : z \in \gamma^*\}$ is finite, and hence

$$|f(z).w^n/z^{n+1}| = |f(z)||z|^{-1}|w/z|^n < \frac{M}{2r}(1/2)^n, \quad \forall z \in \gamma^*.$$

It follows from the Weierstrass M-test that the series

$$\sum_{n=0}^{\infty} \frac{f(z).w^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{f(z)}{z}(w/z)^n = \frac{f(z)}{z}(1-w/z)^{-1} = \frac{f(z)}{z-w}$$

viewed as a function of z, converges uniformly on γ^* to $f(z)/(z-w)$. Thus for all $w \in B(0,r)$ we have

$$I_f(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz\right)w^n,$$

hence $I_f(\gamma, w)$ is given by a power series in $B(0,r)$ (and hence is also holomorphic there) as required.

Finally, if $f = 1$, then since $I_1(\gamma, z) = I(\gamma, z)$ is integer-valued, it follows it must be constant on any connected component of $\mathbb{C}\backslash\gamma^*$ as required. □

Remark 17.8. Note that since the coefficients of a power series centred at a point z_0 are given by its derivatives at that point, the proof above actually also gives formulae for the derivatives of $g(w) = I_f(\gamma, w)$ at z_0:

$$g^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-z_0)^{n+1}}.$$

Remark 17.9. If γ is a closed path then γ^* is compact and hence bounded. Thus there is an $R > 0$ such that the connected set $\mathbb{C}\backslash B(0,R) \cap \gamma^* = \emptyset$. It follows that $\mathbb{C}\backslash\gamma^*$ has exactly one unbounded connected component. Since

$$|\int_\gamma \frac{d\zeta}{\zeta - z}| \leq \ell(\gamma).\sup_{\zeta\in\gamma^*}|1/(\zeta - z)| \to 0$$

as $z \to \infty$ it follows that $I(\gamma, z) = 0$ on the unbounded component of $\mathbb{C}\backslash\gamma^*$.

Definition 17.10. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed path. We say that a point z is in the *inside*[41] of γ if $z \notin \gamma^*$ and $I(\gamma, z) \neq 0$. The previous remark shows that the inside of γ is a union of bounded connected components of $\mathbb{C}\backslash\gamma^*$. (We don't, however, know that the inside of γ is necessarily non-empty.)

Example 17.11. Suppose that $\gamma_1\colon [-\pi,\pi] \to \mathbb{C}$ is given by $\gamma_1 = 1 + e^{it}$ and $\gamma_2\colon [0,2\pi] \to \mathbb{C}$ is given by $\gamma_2(t) = -1 + e^{-it}$. Then if $\gamma = \gamma_1 \star \gamma_2$, γ traverses a figure-of-eight and it is easy to check that the inside of γ is $B(1,1) \cup B(-1,1)$ where $I(\gamma, z) = 1$ for $z \in B(1,1)$ while $I(\gamma, z) = -1$ for $z \in B(-1,1)$.

[41]The term *interior* of γ might be more natural, but we have already used this in the first part of the course to mean something quite different.

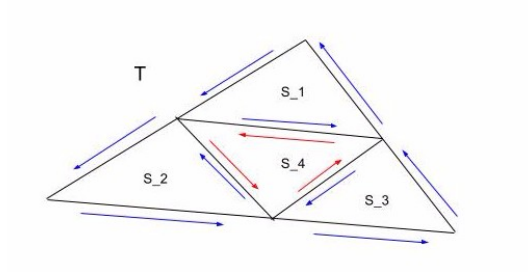

FIGURE 3. Subdivision of a triangle

Remark 17.12. It is a theorem, known as the *Jordan Curve Theorem*, that if $\gamma\colon [0,1] \to \mathbb{C}$ is a simple closed curve, so that $\gamma(t) = \gamma(s)$ if and only if $s = t$ or $s, t \in \{0,1\}$, then $\mathbb{C}\backslash\gamma^*$ is the union of precisely one bounded and one unbounded component, and on the bounded component $I(\gamma, z)$ is either 1 or -1. If $I(\gamma, z) = 1$ for z on the inside of γ we say γ is postively oriented and we say it is negatively oriented if $I(\gamma, z) = -1$ for z on the inside.

18. CAUCHY'S THEOREM

The key insight into the study of holomorphic functions is Cauchy's theorem, which (somewhat informally) states that if $f\colon U \to \mathbb{C}$ is holomorphic and γ is a path in U whose interior lies entirely in U then $\int_\gamma f(z)dz = 0$. It will follow from this and Theorem 16.21 that, at least locally, every holomorphic function has a primitive. The strategy to prove Cauchy's theorem goes as follows: first show it for the simplest closed contours – triangles. Then use this to deduce the existence of a primitive (at least for certain kinds of sufficiently nice open sets U which are called "star-like") and then use Theorem 16.18 to deduce the result for arbitrary paths in such open subsets. We will discuss more general versions of the theorem later, after we have applied Cauchy's theorem for star-like domains to obtain important theorems on the nature of holomorphic functions. First we recall the definition of a triangular path:

Definition 18.1. A *triangle* or *triangular path* T is a path of the form $\gamma_1 \star \gamma_2 \star \gamma_3$ where $\gamma_1(t) = a + t(b-a)$, $\gamma_2(t) = b + t(c-b)$ and $\gamma_3(t) = c + t(a-c)$ where $t \in [0,1]$ and $a, b, c \in \mathbb{C}$. (Note that if $\{a, b, c\}$ are collinear, then T is a degenerate triangle.) That is, T traverses the boundary of the triangle with vertices $a, b, c \in \mathbb{C}$. The solid triangle $\mathcal{T}$ bounded by T is the region

$$\mathcal{T} = \{t_1 a + t_2 b + t_3 c : t_i \in [0,1], \sum_{i=1}^{3} t_i = 1\},$$

with the points in the interior of $\mathcal{T}$ corresponding to the points with $t_i > 0$ for each $i \in \{1,2,3\}$. We will denote by $[a, b]$ the line segment $\{a + t(b-a) : t \in [0,1]\}$, the side of T joining vertex a to vertex b. Whenever it is not evident what the vertices of the triangle T are, we will write $T_{a,b,c}$.

Theorem 18.2. *(Cauchy's theorem for a triangle): Suppose that $U \subseteq \mathbb{C}$ is an open subset and let $T \subseteq U$ be a triangle whose interior is entirely contained in U. Then if $f\colon U \to \mathbb{C}$ is holomorphic we have*

$$\int_T f(z)dz = 0$$

Proof. The proof proceeds using a version of the "divide and conquer" strategy one uses to prove the Bolzano-Weierstrass theorem. Suppose for the sake of contradiction that $\int_T f(z)dz \neq 0$, and let $I = |\int_T f(z)dz| > 0$. We build a sequence of smaller and smaller triangles T^n around which the integral of f is not too small, as follows: Let $T^0 = T$, and suppose that we have constructed T^i for $0 \leq i < k$. Then take the triangle T^{k-1} and join the midpoints of the edges to form four smaller triangles, which we will denote S_i $(1 \leq i \leq 4)$.

Then we have $\int_{T^{k-1}} f(z)dz = \sum_{i=1}^{4} \int_{S_i} f(z)dz$, since the integrals around the interior edges cancel (see Figure 3). In particular, we must have

$$I_k = |\int_{T^{k-1}} f(z)dz| \leq \sum_{i=1}^{4} |\int_{S_i} f(z)dz|,$$

so that for some i we must have $|\int_{S_i} f(z)dz| \geq I_{k-1}/4$. Set T^k to be this triangle S_i. Then by induction we see that $\ell(T^k) = 2^{-k}\ell(T)$ while $I_k \geq 4^{-k}I$.

Now let $\mathscr{T}$ be the solid triangle with boundary T and similarly let $\mathscr{T}^k$ be the solid triangle with boundary T^k. Then we see that $\text{diam}(\mathscr{T}^k) = 2^{-k}\text{diam}(\mathscr{T}) \to 0$, and the sets $\mathscr{T}^k$ are clearly nested. It follows from Lemma 8.6 that there is a unique point z_0 which lies in every $\mathscr{T}^k$. Now by assumption f is holomorphic at z_0, so we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z),$$

where $\psi(z) \to 0 = \psi(z_0)$ as $z \to z_0$. Note that ψ is continuous and hence integrable on all of U. Now since the linear function $z \mapsto f'(z_0)z + f(z_0)$ clearly has a primitive it follows from Theorem 16.18

$$\int_{T^k} f(z)dz = \int_{T^k} (z - z_0)\psi(z)dz$$

Now since z_0 lies in $\mathscr{T}^k$ and z is on the boundary T^k of $\mathscr{T}^k$, we see that $|z-z_0| \leq \text{diam}(\mathscr{T}^k) = 2^{-k}\text{diam}(T)$. Thus if we set $\eta_k = \sup_{z\in T^k} |\psi(z)|$, it follows by the estimation lemma that

$$\begin{aligned} I_k = |\int_{T^k} (z - z_0)\psi(z)dz| &\leq \eta_k.\text{diam}(T^k)\ell(T^k) \\ &= 4^{-k}\eta_k.\text{diam}(T).\ell(T). \end{aligned}$$

But since $\psi(z) \to 0$ as $z \to z_0$, it follows $\eta_k \to 0$ as $k \to \infty$, and hence $4^k I_k \to 0$ as $k \to \infty$. On the other hand, by construction we have $4^k I_k \geq I > 0$, thus we have a contradiction as required. □

Definition 18.3. Let X be a subset in $\mathbb{C}$. We say that X is *convex* if for each $z, w \in U$ the line segment between z and w is contained in X. We say that X is *star-like* if there is a point $z_0 \in X$ such that for every $w \in X$ the line segment $[z_0, w]$ joining z_0 and w lies in X. We will say that X is star-like with respect to z_0 in this case. Thus a convex subset is thus starlike with respect to every point it contains.

Example 18.4. A disk (open or closed) is convex, as is a solid triangle or rectangle. On the other hand a cross, such as $\{0\} \times [-1, 1] \cup [-1, 1] \times \{0\}$ is star-like with respect to the origin, but is not convex.

Theorem 18.5. *(Cauchy's theorem for a star-like domain): Let U be a star-like domain. Then if $f: U \to \mathbb{C}$ is holomorphic and $\gamma: [a, b] \to U$ is a closed path in U we have*

$$\int_\gamma f(z)dz = 0.$$

Proof. The proof proceeds similarly to the proof of Theorem 16.21: by Theorem 16.18 it suffices to show that f has a primitive in U. To show this, let $z_0 \in U$ be a point for which the line segment from z_0 to every $z \in U$ lies in U. Let $\gamma_z = z_0 + t(z - z_0)$ be a parametrization of this curve, and define

$$F(z) = \int_{\gamma_z} f(\zeta)d\zeta.$$

We claim that F is a primitive for f on U. Indeed pick $\epsilon > 0$ such that $B(z,\epsilon) \subseteq U$. Then if $w \in B(z,\epsilon)$ note that the triangle T with vertices z_0, z, w lies entirely in U by the assumption that U is star-like with respect to z_0. It follows from Theorem 18.2 that $\int_T f(\zeta)d\zeta = 0$, and hence if $\eta(t) = w + t(z - w)$ is the straight-line path going from w to z (so that T is the concatenation of γ_w, η and γ_z^-) we have

$$\begin{aligned}
\left|\frac{F(z)-F(w)}{z-w} - f(z)\right| &= \left|\int_\eta \frac{f(\zeta)}{z-w} d\zeta - f(z)\right| \\
&= \left|\int_0^1 f(w+t(z-w))dt - f(z)\right| \\
&= \left|\int_0^1 (f(w+t(z-w)) - f(z)dt\right| \\
&\leq \sup_{t\in[0,1]} |f(w+t(z-w)) - f(z)|,
\end{aligned}$$

which, since f is continuous at w, tends to zero as $w \to z$ so that $F'(z) = f(z)$ as required.

□

Note that our proof of Cauchy's theorem for a star-like domain D proceeded by showing that any holomorphic function on D has a primitive, and hence by the fundamental theorem of calculus its integral around a closed path is zero. This motivates the following definition:

Definition 18.6. We say that a domain $D \subseteq \mathbb{C}$ is *primitive*[42] if any holomorphic function $f\colon D \to \mathbb{C}$ has a primitive in D.

Thus, for example, our proof of Theorem 18.5 shows that all star-like domains are primitive. The following Lemma shows however that we can build many primitive domains which are not star-like.

Lemma 18.7. *Suppose that D_1 and D_2 are primitive domains and $D_1 \cap D_2$ is connected. Then $D_1 \cup D_2$ is primitive.*

Proof. Let $f\colon D_1 \cup D_2 \to \mathbb{C}$ be a holomorphic function. Then $f_{|D_1}$ is a holomorphic function on D_1, and thus it has a primitive $F_1\colon D_1 \to \mathbb{C}$. Similarly $f_{|D_2}$ has a primitive, F_2 say. But then $F_1 - F_2$ has zero derivative on $D_1 \cap D_2$, and since by assumption $D_1 \cap D_2$ is connected (and thus path-connected) it follows $F_1 - F_2$ is constant, c say, on $D_1 \cap D_2$. But then if $F\colon D_1 \cup D_2 \to \mathbb{C}$ is a defined to be F_1 on D_1 and $F_2 + c$ on D_2 it follows that F is a primitive for f on $D_1 \cup D_2$ as required. □

18.1. **Cauchy's Integral Formula.** We are now almost ready to prove one of the most important consequences of Cauchy's theorem – the integral formula. This formula will have incredibly powerful consequences.

Theorem 18.8. *(Cauchy's Integral Formula.) Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function on an open set U which contains the disc $\bar{B}(a,r)$. Then for all $w \in B(a,r)$ we have*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w} dz,$$

where γ is the path $t \mapsto a + re^{2\pi i t}$.

Proof. Fix $w \in B(a,r)$. We use the contours Γ_1 and Γ_2 as shown in Diagram 4 (where Γ_1 follows the direction of the blue arrows, and Γ_2 the directions of the red arrows). These paths join the circular contours $\gamma(a,r)$ and $\gamma(w,\epsilon)^-$ where ϵ is small enough to lie in the interior of $B(a,r)$. By the additivity properties of path integrals, the contributions of the line segments cancel so that

$$\int_{\Gamma_1} \frac{f(z)}{z-w} dz + \int_{\Gamma_2} \frac{f(z)}{z-w} dz = \int_{\gamma(a,r)} \frac{f(z)}{z-w} dz - \int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w} dz.$$

[42] This is *not* standard terminology. The reason for this will become clear later.

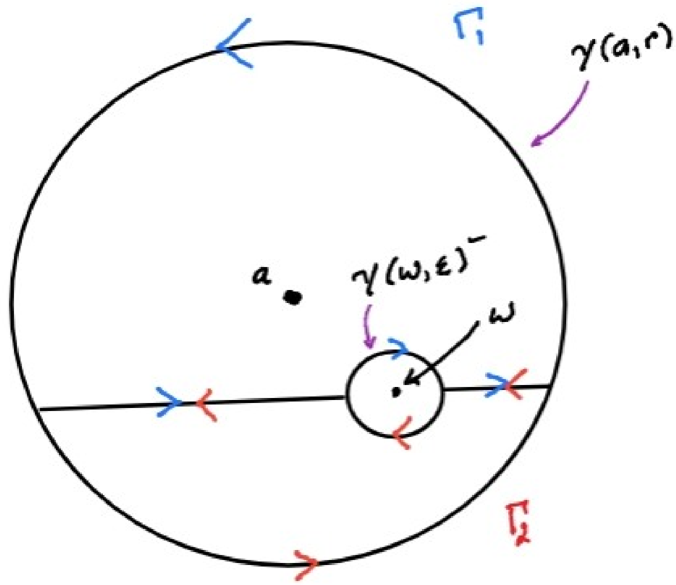

FIGURE 4. Contours for the proof of Theorem 18.8.

On the other hand, each of Γ_1, Γ_2 lies in a primitive domain in which $f/(z-w)$ is holomorphic – indeed by the quotient rule, $f(z)/(z-w)$ is holomophic on $U\backslash\{w\}$ – so each of the integrals on the left-hand side vanish, and hence

$$\frac{1}{2\pi i}\int_{\gamma(a,r)}\frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)}\frac{f(z)}{z-w}dz.$$

Thus we can replace the integral over the circle $\gamma(a,r)$ with an integral over an arbtirary small circle centred at w itself. But for such a small circle,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma(w,\epsilon)}\frac{f(z)}{z-w}dz &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)}\frac{f(z)-f(w)}{z-w}dz + \frac{f(w)}{2\pi i}\int_{\gamma(w,\epsilon)}\frac{dz}{z-w}.\\ &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)}\frac{f(z)-f(w)}{z-w}dz + f(w)I(\gamma(w,\epsilon),w)\\ &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)}\frac{f(z)-f(w)}{z-w}dz + f(w)\end{aligned}$$

But since f is complex differentiable at $z=w$, the term $(f(z)-f(w))/(z-w)$ is bounded as $\epsilon\to 0$, so that by the estimation lemma its integral over $\gamma(w,\epsilon)$ tends to zero. Thus as $\epsilon\to 0$ the path integral around $\gamma(w,\epsilon)$ tends to $f(w)$. But since it is also equal to $(2\pi i)^{-1}\int_{\gamma(a,r)} f(z)/(z-w)dz$, which is independent of ϵ, we conclude that it must in fact be equal to $f(w)$. The result follows.

□

Remark 18.9. The same result holds for any oriented curve γ once we weight the left-hand side by the winding number[43] of a path around the point $w\notin\gamma^*$, provided that f is holomorphic on the inside of γ.

[43]Which, as we used in the proof above, is 1 in the case of a point inside a positively oriented circular path.

18.2. **Applications of the Integral Formula.**

Remark 18.10. Note that Cauchy's integral formula can be interpreted as saying the value of $f(w)$ for w inside the circle is obtained as the "convolution" of f and the function $1/(z-w)$ on the boundary circle. Since the function $1/(z-w)$ is infinitely differentiable one can use this to show that f itself is infinitely differentiable as we will shortly show. If you take the Integral Transforms, you will see convolution play a crucial role in the theory of transforms. In particular, the convolution of two functions often inherits the "good" properties of either. We next show that in fact the formula implies a strong version of Taylor's Theorem.

Corollary 18.11. *If $f: U \to \mathbb{C}$ is holomorphic on an open set U, then for any $z_0 \in U$, the $f(z)$ is equal to its Taylor series at z_0 and the Taylor series converges on any open disk centred at z_0 lying in U. Moreover the derivatives of f at z_0 are given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z-z_0)^{n+1}} dz. \tag{18.1}$$

For any $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ with $z_0 \in B(a,r)$.

Proof. This follows immediately from the proof of Proposition 17.7, and Remark 17.8. The integral formulae of Equation 18.1 for the derivatives of f are also referred to as *Cauchy's Integral Formulae.* $\square$

Definition 18.12. Recall that a function which is locally given by a power series is said to be *analytic.* We have thus shown that any holomorphic function is actually analytic, and from now on we may use the terms interchangeably (as you may notice is common practice in many textbooks).