Metric spaces and complex analysis
Mathematical Institute, University of Oxford
Michaelmas Term 2019

Problem Sheet 6

Throughout this sheet, for $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ we let $\gamma(a, r)$ denote the positively oriented circle centred at a of radius $r > 0$.

1. Suppose that $f: U \to \mathbb{C}$ is a holomorphic function on a domain U.
   (1) Show that, if $\bar{B}(a, r) \subseteq U$ then
$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt.$$
   (2) Suppose that $a \in U$ is such that $|f(a)|$ is a maximum for $|f|: U \to \mathbb{R}$. Show that $|f|$ must in fact be constant near a.
   (3) Deduce that f is constant on all of U.
   [*Hint: Consider the set* $S = \{z \in U : |f(z)| = |f(a)|\}$.

2. (1) Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} \exp(-\frac{1}{x}), & x > 0; \\ 0, & x \leq 0. \end{cases}$$
   is infinitely differentiable everywhere on $\mathbb{R}$ but is not equal to its Taylor series in any open interval centred at 0. (*Hint*: Show by induction that the n-th derivative for $x > 0$ (on the right at $x = 0$) is of the form $f^{(n)}(x) = p_n(1/x)\exp(-1/x)$ where $p_n(x)$ is a polynomial of degree $2n$ and is identically zero for $x \leq 0$.)
   (2) Show that
$$g(x) = \begin{cases} \exp(-z^{-4}), & z \neq 0; \\ 0, & z = 0. \end{cases}$$
   satisfies the Cauchy-Riemann equations at every point in the complex plane, but is not holomorphic on $\mathbb{C}$.

3. Let f be holomorphic on $\mathbb{C}$.
   (1) Prove that f is a polynomial of degree at most k if and only if there exist real constants $M, R > 0$ and an integer k such that
$$|f(z)| \leqslant M\,|z|^k \qquad \text{for } |z| > R.$$
   (2) What holomorphic functions f satisfy $|f(z)| \leqslant |z|^k$ for all $z \in \mathbb{C}$?
   (3) Let $p(z)$ be a polynomial. What holomorphic functions f satisfy $|f(z)| \leqslant |p(z)|$ for all $z \in \mathbb{C}$?

4. Suppose that $f: \mathbb{C} \to \mathbb{C}$ is holomorphic on the whole complex plane (*i.e.* f is an entire function).
   (1) If $f(1/n) = 1/n$ for all $n \in \mathbb{N}$ must $f(z) = z$ for all $z \in \mathbb{C}$?
   (2) If $f(n) = n$ for all $n \in \mathbb{N}$ must $f(z) = z$ for all $z \in \mathbb{C}$?
   (3) Show that there must be some $n \in \mathbb{N}$ such that $f(1/n) \neq 1/(n+1)$.

5. Suppose that $f: \mathbb{C} \to \mathbb{C}$ is an entire function, and for each $z_0 \in \mathbb{C}$ the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ has at least one c_n equals zero. Prove that $f(z)$ is a polynomial.
[*Hint: Note that* $n!c_n = f^{(n)}(z_0)$ *and that* $\mathbb{C}$ *is uncountable.*]

6. Identify the singularities of the following functions. Classify any singularities which are isolated.
$$\frac{1}{e^z - 1}, \qquad \frac{\sin 2\pi z}{z^3(2z-1)}, \qquad \sin\left(\frac{1}{z}\right), \qquad \bar{z}, \qquad \frac{1}{\exp\left(\frac{1}{z}\right) + 2}.$$

7. Let
$$F(z) = \frac{1}{(z-1)^2(z+2)}.$$
Find Laurent expansions for F in
$$A_1 = D(0, 1), \qquad A_2 = \{z : 1 < |z| < 2\}; \qquad A_3 = \left\{z : \sqrt{2} < |z - i| < \sqrt{5}\right\}.$$

8. (*Optional*:) A famous theorem of Weierstrass shows that any continuous function $f\colon [0,1] \to \mathbb{R}$ can be uniformly approximated arbitrarily closely by a polynomial, in that given any $\epsilon > 0$ there is a polynomial $p_\epsilon\colon [0,1] \to \mathbb{R}$ such that $|f(t) - p_\epsilon(t)| < \epsilon$ for all $t \in [0,1]$. If we let $B = \bar{B}(0,1)$ be the closed unit disk in $\mathbb{C}$, is it true that any continuous function $f\colon B \to \mathbb{C}$ on B can be uniformly approximated by polynomials?