One famous application of the Integral formula is known as Liouville's theorem, which will give an easy proof of the Fundamental Theorem of Algebra[44]. We say that a function $f: \mathbb{C} \to \mathbb{C}$ is *entire* if it is complex differentiable on the whole complex plane.

Theorem 18.13. *Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function. If f is bounded then it is constant.*

Proof. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $\gamma_R(t) = Re^{2\pi i t}$ be the circular path centred at the origin with radius R. The for $R > |w|$ the integral formula shows

$$\begin{aligned}
|f(w) - f(0)| &= |\frac{1}{2\pi i}\int_{\gamma_R} f(z)\big(\frac{1}{z-w} - \frac{1}{z}\big)dz| \\
&= \frac{1}{2\pi}|\int_{\gamma_R} \frac{w.f(z)}{z(z-w)}dz| \\
&\leq \frac{2\pi R}{2\pi} \sup_{z:|z|=R} |\frac{w.f(z)}{z(z-w)}| \\
&\leq R.\frac{M|w|}{R.(R-|w|)} = \frac{M|w|}{R-|w|},
\end{aligned}$$

Thus letting $R \to \infty$ we see that $|f(w) - f(0)| = 0$, so that f is constant an required.

□

Theorem 18.14. *Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ is a non-constant polynomial where $a_k \in \mathbb{C}$ and $a_n \neq 0$. Then there is a $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.*

Proof. By rescaling p we may assume that $a_n = 1$. If $p(z) \neq 0$ for all $z \in \mathbb{C}$ it follows that $f(z) = 1/p(z)$ is an entire function (since p is clearly entire). We claim that f is bounded. Indeed since it is continuous it is bounded on any disc $\bar{B}(0,R)$, so it suffices to show that $|f(z)| \to 0$ as $z \to \infty$, that is, to show that $|p(z)| \to \infty$ as $z \to \infty$. But we have

$$|p(z)| = |z^n + \sum_{k=0}^{n-1} a_k z^k| = |z^n|\{|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^n - k}|\} \geq |z^n|.(1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}}).$$

Since $\frac{1}{|z|^m} \to 0$ as $|z| \to \infty$ for any $m \geq 1$ it follows that for sufficiently large $|z|$, say $|z| \geq R$, we will have $1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \geq 1/2$. Thus for $|z| \geq R$ we have $|p(z)| \geq \frac{1}{2}|z|^n$. Since $|z|^n$ clearly tends to infinity as $|z|$ does it follows $|p(z)| \to \infty$ as required. □

Remark 18.15. The crucial point of the above proof is that one term of the polynomial – the leading term in this case– dominates the behaviour of the polynomial for large values of z. All proofs of the fundamental theorem hinge on essentially this point. Note that $p(z_0) = 0$ if and only if $p(z) = (z - z_0)q(z)$ for a polynomial $q(z)$, thus by induction on degree we see that the theorem implies that a polynomial over $\mathbb{C}$ factors into a product of degree one polynomials.

Corollary 18.16. *(Riemann's removable singularity theorem): Suppose that U is an open subset of $\mathbb{C}$ and $z_0 \in U$. If $f: U\backslash\{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then f extends to a holomorphic function on all of U.*

Proof. Define $h(z)$ by

$$h(z) = \begin{cases} (z - z_0)^2 f(z), & z \neq 0; \\ 0, & z = z_0 \end{cases}$$

[44]Which, when it comes down to it, isn't really a theorem in algebra. The most "algebraic" proof of that I know uses Galois theory, which you can learn about in Part B.

The clearly $h(z)$ is holomorphic on $U\backslash\{z_0\}$, using the fact that f and standard rules for complex differentiablility. On the other hand, at $z = z_0$ we see directly that

$$\frac{h(z)-h(z_0)}{z-z_0} = (z-z_0)f(z) \to 0$$

as $z \to z_0$ since f is bounded near z_0 by assumption. It follows that h is in fact holomorphic everywhere in U. But then if we chose $r > 0$ is such that $\bar{B}(z_0, r) \subset U$, then by Corollary 18.11 $h(z)$ is equal to its Taylor series centred at z_0, thus

$$h(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

But since we have $h(z_0) = h'(z_0) = 0$ we see $a_0 = a_1 = 0$, and so $\sum_{k=0}^{\infty} a_{k+2}(z-z_0)^k$ defines a holomorphic function in $B(z_0, r)$. Since this clearly agrees with $f(z)$ on $B(z_0, r)\backslash\{0\}$, we see that by redefining $f(z_0) = a_2$, we can extend f to a holomorphic function on all of U as required. □

We end this section with a kind of converse to Cauchy's theorem:

Theorem 18.17. *(Morera's theorem) Suppose that $f\colon U \to \mathbb{C}$ is a continuous function on an open subset $U \subseteq \mathbb{C}$. If for any closed path $\gamma\colon [a,b] \to U$ we have $\int_\gamma f(z)dz = 0$, then f is holomorphic.*

Proof. By Theorem 16.21 we know that f has a primitive $F\colon U \to \mathbb{C}$. But then F is holomorphic on U and so infinitely differentiable on U, thus in particular $f = F'$ is also holomorphic. □

Remark 18.18. One can prove variants of the above theorem: If U is a star-like domain for example, then our proof of Cauchy's theorem for such domains shows that $f\colon U \to \mathbb{C}$ has a primitive (and hence will be differentiable itself) provided $\int_T f(z)dz = 0$ for every triangle in U. In fact the assumption that $\int_T f(z)dz = 0$ for all triangles whose interior lies in U suffices to imply f is holomorphic for *any* open subset U: To show f is holomorphic on U, it suffices to show that f is holomorphic on $B(a, r)$ for each open disk $B(a, r) \subset U$. But this follows from the above as disks are star-like (in fact convex). It follows that we can characterize the fact that $f\colon U \to \mathbb{C}$ is holomorphic on U by an integral condition: $f\colon U \to \mathbb{C}$ is holomorphic if and only if for all triangles T which bound a solid triangle $\mathscr{T}$ with $\mathscr{T} \subset U$, the integral $\int_T f(z)dz = 0$.

This characterization of the property of being holomorphic has some important consequences. We first need a definition:

Definition 18.19. Let U be an open subset of $\mathbb{C}$. If (f_n) is a sequence of functions defined on U, we say $f_n \to f$ *uniformly on compacts* if for every compact subset K of U, the sequence $(f_{n|K})$ converges uniformly to $f_{|K}$. Note that in this case f is continuous if the f_n are: Indeed to see that f is continuous at $a \in U$, note that since U is open, there is some $r > 0$ with $B(a, r) \subseteq U$. But then $K = \bar{B}(a, r/2) \subseteq U$ and $f_n \to f$ uniformly on K, whence f is continuous on K, and so certainly it is continuous at a.

Example 18.20. Convergence of power series $f(z) = \sum_{k=0}^{\infty} a_n z^n$ is a basic example of convergence on compacts: if R is the radius of convergences of $f(z)$ the partial sums $s_n(z)$ of the power series $B(0, R)$ converge uniformly on compacts in $B(0, R)$. The convergence is *not* necessarily uniform on $B(0, R)$, as the example $f(z) = \sum_{n=0}^{\infty} z^n$ shows. Nevertheless, since $B(0, R) = \bigcup_{r<R} \bar{B}(0, r)$ is the union of its compact subsets, many of the good properties of the polynomial functions $s_n(z)$ are inherited by the power series because the convergence is uniform on compact subsets.

Proposition 18.21. *Suppose that U is a domain and the sequence of holomorphic functions $f_n\colon U \to \mathbb{C}$ converges to $f\colon U \to \mathbb{C}$ uniformly on compacts in U. Then f is holomorphic.*

Proof. Note by the above that f is continuous on U. Since the property of being holomorphic is local, it suffices to show for each $w \in U$ that there is a ball $B(w, r) \subseteq U$ within which f is holomorphic. Since U is open, for any such w we may certainly find $r > 0$ such that $B(w, r) \subseteq U$. Then as $B(w, r)$ is convex, Cauchy's theorem for a star-like domain shows that for every closed path $\gamma\colon [a,b] \to B(w,r)$ whose image lies in $B(w,r)$ we have $\int_\gamma f_n(z)dz = 0$ for all $n \in \mathbb{N}$.

But $\gamma^* = \gamma([a,b])$ is a compact subset of U, hence $f_n \to f$ uniformly on γ^*. It follows that

$$0 = \int_\gamma f_n(z)dz \to \int_\gamma f(z)dz,$$

so that the integral of f around any closed path in $B(w,r)$ is zero. But then Theorem 16.21 shows that f has a primitive F on $B(w,r)$. But we have seen that any holomorphic function is in fact infinitely differentiable, so it follows that F, and hence f is infinitely differentiable on $B(w,r)$ as required.

□

Often functions on the complex plane are defined in terms of integrals. It is thus useful to have a criterion by which one can check if such a function is holomorphic. The following theorem gives such a criterion.

Theorem 18.22. *Let U be an open subset of $\mathbb{C}$ and suppose that $F\colon U \times [a,b]$ is a function satisfying*

(1) *The function $z \mapsto F(z,s)$ is holomorphic in z for each $s \in [a,b]$.*
(2) *F is continuous on $U \times [a,b]$*

Then the function $f\colon U \to \mathbb{C}$ defined by

$$f(z) = \int_a^b F(z,s)ds$$

is holomorphic.

Proof. Changing variables we may assume that $[a,b] = [0,1]$ (explicitly, one replaces s by $(s-a)/(b-a)$). By Theorem 18.21 it is enough to show that we may find a sequence of holomorphic functions $f_n(z)$ which converge of $f(z)$ uniformly on compact subsets of U. To find such a sequence, recall from Prelims Analysis that the Riemann integral of a continuous function is equal to the limit of its Riemann sums as the mesh of the partition used for the sum tends to zero. Using the partition $x_i = i/n$ for $0 \leq i \leq n$ evaluating at the right-most end-point of each interval, we see that

$$f_n(z) = \frac{1}{n}\sum_{i=1}^{n} F(z, i/n),$$

is a Riemann sum for the integral $\int_0^1 F(z,s)ds$, hence as $n \to \infty$ we have $f_n(z) \to f(z)$ for each $z \in U$, *i.e.* the sequence (f_n) converges pointwise to f on all of U. To complete the proof of the theorem it thus suffices to check that $f_n \to f$ as $n \to \infty$ uniformly on compact subsets of U. But if $K \subseteq U$ is compact, then since F is clearly continuous on the compact set $K \times [0,1]$, it is uniformly continuous there, hence, given any $\epsilon > 0$, there is a $\delta > 0$ such that $|F(z,s) - F(z,t)| < \epsilon$ for all $z \in \bar{B}(a,\rho)$ and $s,t \in [0,1]$ with $|s-t| < \delta$. But then if $n > \delta^{-1}$ we have for all $z \in K$

$$\begin{aligned}
|f(z) - f_n(z)| &= |\int_0^1 F(z,s)dz - \frac{1}{n}\sum_{i=1}^{n} F(z,i/n)| \\
&= \left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n} \big(F(z,s) - F(z,i/n)\big)ds\right| \\
&\leq \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n} |F(z,s) - F(z,i/n)|ds \\
&< \sum_{i=1}^{n} \epsilon/n = \epsilon.
\end{aligned}$$

Thus $f_n(z)$ tends to $f(z)$ uniformly on K as required. □

Example 18.23. If f is any continuous function on $[0,1]$, then the previous theorem shows that the function $f(z) = \int_0^1 e^{isz} f(s)ds$ is holomorphic in z, since clearly $F(z,s) = e^{isz} f(z)$ is continuous as a function on $\mathbb{C} \times [0,1]$ and, for fixed $s \in [0,1]$, F is holomorphic as a function of z. Integrals of this nature (though perhaps over the whole real line or the positive real axis) arise frequently in many parts of mathematics, as you can learn more about in the optional course on Integral Transforms.

Remark 18.24. Another way to prove the theorem is to use Morera's theorem directly: if $\gamma\colon [0,1] \to \mathbb{C}$ is a closed path in $B(a,r)$, then we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \Big(\int_0^1 F(z,s)ds\Big)dz\\ &= \int_0^1 \Big(\int_\gamma F(z,s)dz\Big)ds = 0,\end{aligned}$$

where in the first line we interchanged the order of integration, and in the second we used the fact that $F(z,s)$ is holomorphic in z and Cauchy's theorem for a disk. To make this completely rigorous however, one has to justify the interchange of the orders of integration. Next term's course on Integration proves a very general result of this form known as Fubini's theorem, but for continous functions on compact subets of $\mathbb{R}^n$ one can give more elementary arguments by showing any such function is a uniform limit of linear combinations of indicator functions of "boxes" – the higher dimensional analogues of step functions – and the elementary fact that the interchange of the order of integration for indicator functions of boxes holds trivially.

19. The identity theorem, isolated zeros and singularities

The fact that any complex differentiable function is in fact analytic has some very surprising consequences – the most striking of which is perhaps captured by the "Identity theorem". This says that if f, g are two holomorphic functions defined on a domain U and we let $S = \{z \in U : f(z) = g(z)\}$ be the locus on which they are equal, then if S has a limit point in U it must actually be all of U. Thus for example if there is a disk $B(a,r) \subseteq U$ on which f and g agree (not matter how small r is), then in fact they are equal on all of U! The key to the proof of the Identity theorem is the following result on the zeros of a holomorphic function:

Proposition 19.1. *Let U be an open set and suppose that $g\colon U \to \mathbb{C}$ is holomorphic on U. Let $S = \{z \in U : g(z) = 0\}$. If $z_0 \in S$ then either z_0 is isolated in S (so that g is non-zero in some disk about z_0 except at z_0 itself) or $g = 0$ on a neighbourhood of z_0. In the former case there is a unique integer $k > 0$ and holomorphic function g_1 such that $g(z) = (z - z_0)^k g_1(z)$ where $g_1(z_0) \neq 0$.*

Proof. Pick any $z_0 \in U$ with $g(z_0) = 0$. Since g is analytic at z_0, if we pick $r > 0$ such that $\bar{B}(z_0, r) \subseteq U$, then we may write

$$g(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k,$$

for all $z \in B(z_0, r) \subseteq U$, where the coeficients c_k are given as in Theorem 18.11. Now if $c_k = 0$ for all k, it follows that $g(z) = 0$ for all $z \in B(0,r)$. Otherwise, we set $k = \min\{n \in \mathbb{N} : c_n \neq 0\}$ (where since $g(z_0) = 0$ we have $c_0 = 0$ so that $k \geq 1$). Then if we let $g_1(z) = (z - z_0)^{-k} g(z)$, clearly $g_1(z)$ is holomorphic on $U \backslash \{z_0\}$, but since in $B(z_0, r)$ we have we have $g_1(z) = \sum_{n=0}^{\infty} c_{k+n} (z - z_0)^n$, it follows if we set $g_1(z_0) = c_k \neq 0$ then g_1 becomes a holomorphic function on all of U. Since g_1 is continuous at z_0 and $g_1(z_0) \neq 0$, there is an $\epsilon > 0$ such that $g_1(z) \neq 0$ for all $z \in B(z_0, \epsilon)$. But $(z - z_0)^k$ vanishes only at z_0, hence it follows that $g(z) = (z - z_0)^k g_1(z)$ is non-zero on $B(a,\epsilon) \backslash \{z_0\}$, so that z_0 is isolated.

Finally, to see that k is unique, suppose that $g(z) = (z - z_0)^k g_1(z) = (z - z_0)^l g_2(z)$ say with $g_1(z_0)$ and $g_2(z_0)$ both nonzero. If $k < l$ then $g(z)/(z - z_0)^k = (z - z_0)^{l-k} g_2(z)$ for all $z \neq z_0$, hence as $z \to z_0$ we have

$g(z)/(z-z_0)^k \to 0$, which contradicts the assumption that $g_1(z) \neq 0$. By symmetry we also cannot have $k > l$ so $k = l$ as required. ☐

Remark 19.2. The integer k in the previous proposition is called the *multiplicity* of the zero of g at $z = z_0$ (or sometimes the *order of vanishing*).

Theorem 19.3. *(Identity theorem): Let U be a domain and suppose that f_1, f_1 are holomorphic functions defined on U. Then if $S = \{z \in U : f_1(z) = f_2(z)\}$ has a limit point in U, we must have $S = U$, that is $f_1(z) = f_2(z)$ for all $z \in U$.*

Proof. Let $g = f_1 - f_2$, so that $S = g^{-1}(\{0\})$. We must show that if S has a limit point then $S = U$. Since g is clearly holomorphic in U, by Proposition 19.1 we see that if $z_0 \in S$ then either z_0 is an isolated point of S or it lies in an open ball contained in S. It follows that $S = V \cup T$ where $T = \{z \in S : z \text{ is isolated}\}$ and $V = \text{int}(S)$ is open. But since g is continuous, $S = g^{-1}(\{0\})$ is closed in U, thus $V \cup T$ is closed, and so $\text{Cl}_U(V)$, the closure[45] of V in U, lies in $V \cup T$. However, by definition, no limit point of V can lie in T so that $\text{Cl}_U(V) = V$, and thus V is open and closed in U. Since U is connected, it follows that $V = \emptyset$ or $V = U$. In the former case, all the zeros of g are isolated so that $S' = T' = \emptyset$ and S has no limit points. In the latter case, $V = S = U$ as required.

☐

Remark 19.4. The requirement in the theorem that S have a limit point *lying in U* is essential: If we take $U = \mathbb{C}\backslash\{0\}$ and $f_1 = \exp(1/z) - 1$ and $f_2 = 0$, then the set S is just the points where f_1 vanishes on U. Now the zeros of f_1 have a limit point at $0 \notin U$ since $f(1/(2\pi i n)) = 0$ for all $n \in \mathbb{N}$, but certainly f_1 is not identically zero on U!

We now wish to study singularities of holomorphic functions. The key result here is Riemann's removable singularity theorem, Corollary 18.16.

Definition 19.5. If U is an open set in $\mathbb{C}$ and $z_0 \in U$, we say that a function $f : U\backslash\{z_0\} \to \mathbb{C}$ has an *isolated singularity* at z_0 if it is holomorphic on $B(z_0, r)\backslash\{z_0\}$ for some $r > 0$.

Suppose that z_0 is an isolated singularity of f. If f is bounded near z_0 we say that f has a *removable singularity* at z_0, since by Corollary 18.16 it can be extended to a holomorphic function at z_0. If f is not bounded near z_0, but the function $1/f(z)$ has a removable singularity at z_0, that is, $1/f(z)$ extends to a holomorphic function on all of $B(z_0, r)$, then we say that f has a *pole* at z_0. By Proposition 19.1 we may write $(1/f)(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$ and $m \in \mathbb{Z}_{>0}$. (Note that the extension of $1/f$ to z_0 must vanish there, as otherwise f would be bounded near z_0.) We say that m is the *order* of the pole of f at z_0. In this case we have $f(z) = (z - z_0)^{-m}.(1/g)$ near z_0, where $1/g$ is holomorphic near z_0 since $g(z_0) \neq 0$. If $m = 1$ we say that f has a *simple pole* at z_0.

Finally, if f has an isolated singularity at z_0 which is not removable nor a pole, we say that z_0 is an *essential singularity*.

Lemma 19.6. *Let f be a holomorphic function with a pole of order m at z_0. Then there is an $r > 0$ such that for all $z \in B(z_0, r)\backslash\{z_0\}$ we have*

$$f(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

Proof. As we have already seen, we may write $f(z) = (z - z_0)^{-m} h(z)$ where m is the order of the pole of f at z_0 and $h(z)$ is holomorphic and non-vanishing at z_0. The claim follows since, near z_0, $h(z)$ is equal to its Taylor series at z_0, and multiplying this by $(z - z_0)^{-m}$ gives a series of the required form for $f(z)$. ☐

Definition 19.7. The series $\sum_{n \geq -m} c_n (z - z_0)^n$ is called the *Laurent series* for f at z_0. We will show later that if f has an isolated essential singularity it still has a Laurent series expansion, but the series is then involves infinitely many positive and negative powers of $(z - z_0)$.

[45] I use the notation $\text{Cl}_U(V)$, as opposed to $\bar{V}$, to emphasize that I mean the closure of V in U, not in $\mathbb{C}$, that is, $\text{Cl}_U(V)$ is equal to the union of V with the limits points of V which lie in U.

A function on an open set U which has only isolated singularities all of which are poles is called a *meromorphic* function on U. (Thus, strictly speaking, it is a function only defined on the complement of the poles in U.)

Lemma 19.8. *Suppose that f has an isolated singularity at a point z_0. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole of f then $1/f(z) = (z-z_0)^k g(z)$ where $g(z_0) \neq 0$ and $k > 0$. But then for $z \neq z_0$ we have $f(z) = (z-z_0)^{-k}(1/g(z))$, and since $g(z_0) \neq 0$, $1/g(z)$ is bounded away from 0 near z_0, while $|(z-z_0)^{-k}| \to \infty$ as $z \to z_0$, so $|f(z)| \to \infty$ as $z \to z_0$ as required.

On the other hand, if $|f(z)| \to \infty$ as $z \to z_0$, then $1/f(z) \to 0$ as $z \to z_0$, so that $1/f(z)$ has a removable singularity and f has a pole at z_0. □

Remark 19.9. The previous Lemma can be rephrased to say that f has a pole at z_0 precisely when f extends to a continuous function $f\colon U \to \mathbb{C}_\infty$ with $f(z_0) = \infty$. Moreover, you can check from Definition 13.6 that in this case, the extension is actually holomorphic. Thus the Riemann sphere allows us to put holomorphic and meromorphic functions on the same footing.

The case where f has an essential singularity is more complicated. We prove that near an isolated singularity the values of a holomorphic function are dense:

Theorem 19.10. *(Casorati-Weierstrass): Let U be an open subset of $\mathbb{C}$ and let $a \in U$. Suppose that $f\colon U\backslash\{a\} \to \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a. Then for all $\rho > 0$ with $B(a,\rho) \subseteq U$, the set $f(B(a,\rho)\backslash\{a\})$ is dense in $\mathbb{C}$, that is, the closure of $f(B(a,\rho)\backslash\{a\})$ is all of $\mathbb{C}$.*

Proof. Suppose, for the sake of a contradiction, that there is some $\rho > 0$ such that $z_0 \in \mathbb{C}$ is not a limit point of $f(B(a,\rho)\backslash\{a\})$. Then the function $g(z) = 1/(f(z)-z_0)$ is bounded and non-vanishing on $B(a,\rho)\backslash\{a\}$, and hence by Riemann's removable singularity theorem, it extends to a holomorphic function on all of $B(a,\rho)$. But then $f(z) = z_0 + 1/g(z)$ has at most a pole at a which is a contradiction. □

Remark 19.11. In fact much more is true: Picard showed that if f has an isolated essential singularity at z_0 then in any open disk about z_0 the function f takes every complex value infinitely often with at most one exception. The example of the function $f(z) = \exp(1/z)$, which has an essential singularity at $z = 0$ shows that this result is best possible, since $f(z) \neq 0$ for all $z \neq 0$.

19.1. **Principal parts.**

Definition 19.12. Recall that by Lemma 19.6 if a function f has a pole of order k at z_0 then near z_0 we may write

$$f(z) = \sum_{n \geq -k} c_n (z-z_0)^n.$$

The function $\sum_{n=-k}^{-1} c_n(z-z_0)^n$ is called the *principal part* of f at z_0, and we will denote it by $P_{z_0}(f)$. It is a rational function which is holomorphic on $\mathbb{C}\backslash\{z_0\}$. Note that $f - P_{z_0}(f)$ is holomorphic at z_0 (and also holomorphic wherever f is). The *residue* of f at z_0 is defined to be the coefficient c_{-1} and denoted $\mathrm{Res}_{z_0}(f)$.

The reason for introducing these definitions is the following: Suppose that $f\colon U \to \mathbb{C}_\infty$ is a meromorphic function with poles at a finite set $S \subseteq U$. Then for each $z_0 \in S$ we have the principal part $P_{z_0}(f)$ of f at z_0, a rational function which is holomorphic everywhere on $\mathbb{C}\backslash\{z_0\}$. The difference

$$g(z) = f(z) - \sum_{z_0 \in S} P_{z_0}(f),$$

is holomorphic on all of U (away from S the is clear because each term is, at $z_0 \in S$ the terms $P_s(f)$ for $s \in S\backslash\{z_0\}$ are all holomorphic, while $f(z) - P_{z_0}(f)$ is holomorphic at z_0 by the definition of $P_{z_0}(f)$). Thus if U is starlike and $\gamma\colon [0,1] \to U$ is any closed path in U with $\gamma^* \cap S = \emptyset$, we have

$$\int_\gamma f(z)dz = \int_\gamma g(z)dz + \sum_{z_0 \in S} \int_\gamma P_{z_0}(f)dz = \sum_{z_0 \in S} \int_\gamma P_{z_0}(f)dz.$$

The most important term in the principal part $P_{z_0}(f)$ is the term $c_{-1}/(z-z_0)$. This is because every other term has a primitive on $\mathbb{C}\backslash\{z_0\}$, hence by the Fundamental Theorem of Calculus it is the only part which contributes to the integral of $P_{z_0}(f)$ around the closed path γ. Combining these observations we see that

$$\int_\gamma f(z)dz = \sum_{z_0 \in S} \text{Res}_{z_0}(f) \int_\gamma \frac{dz}{z-z_0} = 2\pi i \sum_{z_0 \in S} \text{Res}_{z_0}(f).I(\gamma, z_0),$$

where $I(\gamma, z_0)$ denotes the winding number of γ about the pole z_0. This is the *residue theorem* for meromorphic functions on a starlike domain. We will shortly generalize it.

Lemma 19.13. *Suppose that f has a pole of order m at z_0, then*

$$Res_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z))$$

Proof. Since f has a pole of order m at z_0 we have $f(z) = \sum_{n \geq -m} c_n (z-z_0)^n$ for z sufficiently close to z_0. Thus

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \ldots + c_{-1}(z-z_0)^{m-1} + \ldots$$

and the result follows from the formula for the derivatives of a power series. □

Remark 19.14. The last lemma is perhaps most useful in the case where the pole is simple, since in that case no derivatives need to be computed. In fact there is a special case which is worth emphasizing: Suppose that $f = g/h$ is a ratio of two holomorphic functions defined on a domain $U \subseteq \mathbb{C}$, where h is non-constant. Then f is meromorphic with poles at the zeros[46] of h. In particular, if h has a simple zero at z_0 and g is non-vanishing there, then f correspondingly has a simple pole at z_0. Since the zero of h is simple at z_0, we must have $h'(z_0) \neq 0$, and hence by the previous result

$$\text{Res}_{z_0}(f) = \lim_{z \to z_0} \frac{g(z)(z-z_0)}{h(z)} = \lim_{z \to z_0} g(z). \lim_{z \to z_0} \frac{z-z_0}{h(z)-h(z_0)} = g(z_0)/h'(z_0)$$

where the last equality holds by standard Algebra of Limits results.

20. Homotopies, simply-connected domains and Cauchy's theorem

A crucial point in our proof of Cauchy's theorem for a triangle was that the interior of the triangle was entirely contained in the open set on which our holomorphic function f was defined. In general however, given a closed curve, it is not always easy to say what we mean by the "interior" of the curve. In fact there is a famous theorem, known as the Jordan Curve Theorem, which resolves this problem, but to prove it would take us too far afield. Instead we will take a slightly different strategy: in fact we will take two different approaches: the first using the notion of homotopy and the second using the winding number. For the homotopy approach, rather than focusing only on closed curves and their "interiors" we consider arbitrary curves and study what it means to deform one to another.

Definition 20.1. Suppose that U is an open set in $\mathbb{C}$ and $a, b \in U$. If $\eta\colon [0,1] \to U$ and $\gamma\colon [0,1] \to U$ are paths in U such that $\gamma(0) = \eta(0) = a$ and $\gamma(1) = \eta(1) = b$, then we say that γ and η are *homotopic* in U if there is a continuous function $h\colon [0,1] \times [0,1] \to U$ such that

$$h(0,s) = a, \quad h(1,s) = b$$
$$h(t,0) = \gamma(t), \quad h(t,1) = \eta(t).$$

[46]Strictly speaking, the poles of f form a subset of the zeros of h, since if g also vanishes at a point z_0, then f may have a removable singularity at z_0.

One should think of h as a family of paths in U indexed by the second variable s which continuously deform γ into η.

A special case of the above definition is when $a = b$ and γ and η are closed paths. In this case there is a constant path $c_a\colon [0,1] \to U$ going from a to $b = a$ which is simply given by $c_a(t) = a$ for all $t \in [0,1]$. We say a closed path starting and ending at a point $a \in U$ is *null homotopic* if it is homotopic to the constant path c_a. One can show that the relation "γ is homotopic to η" is an equivalence relation, so that any path γ between a and b belongs to a unique equivalence class, known as its homotopy class.

Definition 20.2. Suppose that U is a domain in $\mathbb{C}$. We say that U is *simply connected* if for every $a, b \in U$, any two paths from a to b are homotopic in U.

Lemma 20.3. *Let U be a convex open set in $\mathbb{C}$. Then U is simply connected. Moreover if U_1 and U_2 are homeomorphic, then U_1 is simply connected if and only if U_2 is.*

Proof. Suppose that $\gamma\colon [0,1] \to U$ and $\eta\colon [0,1] \to U$ are paths starting and ending at a and b respectively for some $a, b \in U$. Then for $(s,t) \in [0,1] \times [0,1]$ let

$$h(t,s) = (1-s)\gamma(t) + s\eta(t)$$

It is clear that h is continuous and one readily checks that h gives the required homotopy. For the moreover part, if $f\colon U_1 \to U_2$ is a homeomorphism then it is clear that f induces a bijection between continuous paths in U_1 to those in U_2 and also homotopies in U_1 to those in U_2, so the claim follows. □

Remark 20.4. (*Non-examinable*) In fact, with a bit more work, one can show that any starlike domain D is also simply-connected. The key is to show that a domain is simply-connected if all closed paths starting and ending at a given point $z_0 \in D$ are null-homotopic. If D is star-like with respect to $z_0 \in D$, then if $\gamma\colon [0,1] \to D$ is a closed path with $\gamma(0) = \gamma(1) = z_0$, it follows $h(s,t) = z_0 + s(\gamma(t) - z_0)$ gives a homotopy between γ and the constant path c_{z_0}.

Thus we see that we already know many examples of simply connected domains in the plane, such as disks, ellipsoids, half-planes. The second part of the above lemma also allows us to produce non-convex examples:

Example 20.5. Consider the domain

$$D_{\eta,\epsilon} = \{z \in \mathbb{C} : z = re^{i\theta} : \eta < r < 1, 0 < \theta < 2\pi(1-\epsilon)\},$$

where $0 < \eta, \epsilon < 1/10$ say, then $D_{\eta,\epsilon}$ is clearly not convex, but it is the image of the convex set $(0,1) \times (0,1-\epsilon)$ under the map $(r,\theta) \mapsto re^{2\pi i\theta}$. Since this map has a continuous (and even differentiable) inverse, it follows $D_{\eta,\epsilon}$ is simply-connected. When η and ϵ are small, the boundary of this set, oriented anti-clockwise, is a version of what is called a *key-hole contour*.

We are now ready to state our extension of Cauchy's theorem. The proof is given in the Appendices.

Theorem 20.6. *Let U be a domain in $\mathbb{C}$ and $a, b \in U$. Suppose that γ and η are paths from a to b which are homotopic in U and $f\colon U \to \mathbb{C}$ is a holomorphic function. Then*

$$\int_\gamma f(z)dz = \int_\eta f(z)dz.$$

Remark 20.7. Notice that this theorem is really more general than the previous versions of Cauchy's theorem we have seen – in the case where a holomorphic function $f\colon U \to \mathbb{C}$ has a primitive the conclusion of the previous theorem is of course obvious from the Fundamental theorem of Calculus[47], and our previous formulations of Cauchy's theorem were proved by producing a primitive for f on U. One significance of the homotopy form of Cauchy's theorem is that it applies to domains U even when there is no primitive for f on U.

[47]Indeed the hypothesis that the paths γ and η are homotopic is irrelevant when f has a primitive on U.

Theorem 20.8. *Suppose that U is a simply-connected domain, let $a, b \in U$, and let $f: U \to \mathbb{C}$ be a holomorphic function on U. Then if γ_1, γ_2 are paths from a to b we have*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

In particular, if γ is a closed oriented curve we have $\int_\gamma f(z)dz = 0$, and hence any holomorphic function on U has a primitive.

Proof. Since U is simply-connected, any two paths from from a to b are homotopic, so we can apply Theorem 20.6. For the last part, in a simply-connected domain any closed path $\gamma: [0,1] \to U$, with $\gamma(0) = \gamma(1) = a$ say, is homotopic to the constant path $c_a(t) = a$, and hence $\int_\gamma f(z)dz = \int_{c_a} f(z)dz = 0$. The final assertion then follows from the Theorem 16.21. □

Example 20.9. If $U \subseteq \mathbb{C}\backslash\{0\}$ is simply-connected, the previous theorem shows that there is a holomorphic branch of $[\mathrm{Log}(z)]$ defined on all of U (since any primitive for $f(z) = 1/z$ will be such a branch).

Remark 20.10. Recall that in Definition 18.6 we called a domain D in the complex plane *primtive* if every holomorphic function $f: D \to \mathbb{C}$ on it had a primitive. Theorem 20.8 shows that any simply-connected domain is primitive. In fact the converse is also true – any primitive domain is necessarily simply-connected. Thus the term "primitive domain" is in fact another name for a simply-connected domain.

The definition of winding number allows us to give another version of Cauchy's integral formula (sometimes called the *winding number* or *homology* form of Cauchy's theorem).

Theorem 20.11. *Let $f: U \to \mathbb{C}$ be a holomorphic function and let $\gamma: [0,1] \to U$ be a closed path whose inside lies entirely in U, that is $I(\gamma, z) = 0$ for all $z \notin U$. Then we have, for all $z \in U\backslash\gamma^*$,*

$$\int_\gamma f(\zeta)d\zeta = 0; \quad \int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = 2\pi i I(\gamma, z)f(z).$$

Moreover, if U is simply-connected and $\gamma: [a,b] \to U$ is any closed path, then $I(\gamma, z) = 0$ for any $z \notin U$, so the above identities hold for all closed paths in such U.

Remark 20.12. The "moreover" statement in fact just uses the fact that a simply-connected domain is primitive: if D is a domain and $w \notin D$, then the function $1/(z - w)$ is holomorphic on all of D, and hence has a primitive on D. It follows $I(\gamma, w) = 0$ for any path γ with $\gamma^* \subseteq D$.

Remark 20.13. This version of Cauchy's theorem has a natural extension: instead of integrating over a single closed path, one can integrate over formal sums of closed paths, which are known as *cycles*: if $a \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_k$ are closed paths and $a_1, \dots, a_k$ are complex numbers (we will usually only consider the case where they are integers) then we define the integral around the formal sum $\Gamma = \sum_{i=1}^k a_i\gamma_i$ of a function f to be

$$\int_\Gamma f(z)dz = \sum_{i=1}^k a_i \int_{\gamma_i} f(z)dz.$$

Since the winding number can be expressed as an integral, this also gives a natural defintion of the winding number for such Γ: explicitly $I(\Gamma, z) = \sum_{i=1}^k a_i I(\gamma_i, z)$. If we write $\Gamma^* = \gamma_1^* \cup \ldots \cup \gamma_k^*$ then $I(\Gamma, z)$ is defined for all $z \notin \Gamma^*$. The winding number version Cauchy's theorem then holds (with the same proof) for cycles in an open set U, where we define the inside of a cycle to be the set of $z \in \mathbb{C}$ for which $I(\Gamma, z) \neq 0$.

Note that if z is inside Γ then it must be the case that z is inside some γ_i, but the converse is not necessarily the case: it may be that z lies inside some of the γ_i but does not lie inside Γ. One natural way in which cycles arise are as the boundaries of an open subsets of the plane: if Ω is an domain in the plane, then $\partial\Omega$, the boundary of Ω is often a *union* of curves rather than a single curve[48]. For example if

[48]Of course in general the boundary of an open set need not be so nice as to be a union of curves at all.

$r < R$ then $\Omega = B(0,R)\backslash\bar{B}(0,r)$ has a boundary which is a union of two concentric circles. If these circles are oriented correctly, then the "inside" of the cycle Γ which they form is precisely Ω (see the discussion of Laurent series below for more details). Thus the origin, although inside each of the circles $\gamma(0,r)$ and $\gamma(0,R)$, is not inside Γ. The cycles version of Cauchy's theorem is thus closest to Green's theorem in multivariable calculus.

As a first application of this new form of Cauchy's theorem, we establish the *Laurent expansion* of a function which is holomorphic in an annulus. This is a generalization of Taylor's theorem, and we already saw it in the special case of a function with a pole singularity.

Definition 20.14. Let $0 < r < R$ be real numbers and let $z_0 \in \mathbb{C}$. An open *annulus* is a set

$$A = A(r,R,z_0) = B(z_0,R)\backslash\bar{B}(z_0,r) = \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$

If we write (for $s > 0$) $\gamma(z_0,s)$ for the closed path $t \mapsto z_0 + se^{2\pi i t}$ then notice that the inside of the cycle $\Gamma_{r,R,z_0} = \gamma(z_0,R) - \gamma(z_0,r)$ is precisely A, since for any s, $I(\gamma(z_0,s),z)$ is 1 precisely if $z \in B(z_0,s)$ and 0 otherwise.

Theorem 20.15. *Suppose that* $0 < r < R$ *and* $A = A(r,R,z_0)$ *is an annulus centred at* z_0*. If* $f\colon U \to \mathbb{C}$ *is holomorphic on an open set* U *which contains* $\bar{A}$*, then there exist* $c_n \in \mathbb{C}$ *such that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \forall z \in A.$$

Moreover, the c_n *are unique and are given by the following formulae:*

$$c_n = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where $s \in [r,R]$ *and for any* $s > 0$ *we set* $\gamma_s(t) = z_0 + se^{2\pi i t}$.

Proof. By translation we may assume that $z_0 = 0$. Since A is the inside of the cycle Γ_{r,R,z_0} it follows from the winding number form of Cauchy's integral formula that for $w \in A$ we have

$$2\pi i f(w) = \int_{\gamma_R} \frac{f(z)}{z - w} dz - \int_{\gamma_r} \frac{f(z)}{z - w} dz$$

But now the result follows in the same way as we showed holomorphic functions were analytic: if we fix w, then, for $|w| < |z|$ we have $\frac{1}{z-w} = \sum_{n=0}^{\infty} w^n/z^{n+1}$, converging uniformly in z in $|z| > |w| + \epsilon$ for any $\epsilon > 0$. It follows that

$$\int_{\gamma_R} \frac{f(z)}{z - w} dz = \int_{\gamma_R} \sum_{n=0}^{\infty} \frac{f(z) w^n}{z^{n+1}} dz = \sum_{n \geq 0} \left(\int_{\gamma_R} \frac{f(z)}{z^{n+1}} dz \right) w^n.$$

for all $w \in A$. Similarly since for $|z| < |w|$ we have[49] $\frac{1}{w-z} = \sum_{n \geq 0} z^n/w^{n+1} = \sum_{n=-1}^{-\infty} w^n/z^{n+1}$, again converging uniformly on $|z|$ when $|z| < |w| - \epsilon$ for $\epsilon > 0$, we see that

$$\int_{\gamma_r} \frac{f(z)}{w - z} dz = \int_{\gamma_r} \sum_{n=-1}^{-\infty} f(z) w^n/z^{n+1} dz = \sum_{n=-1}^{-\infty} \big(\int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz \big) w^n.$$

Thus taking $(c_n)_{n \in \mathbb{Z}}$ as in the statement of the theorem, we see that

$$f(w) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z - w} dz - \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z - w} dz = \sum_{n \in \mathbb{Z}} c_n z^n,$$

as required. To see that the c_n are unique, one checks using uniform convergence that if $\sum_{n \in \mathbb{Z}} d_n z^n$ is any series expansion for $f(z)$ on A, then the d_n must be given by the integral formulae above.

[49] Note the sign change.

Finally, to see that the c_n can be computed using any circular contour γ_s, note that if $r \le s_1 < s_2 \le R$ then $f/(z-z_0)^{n+1}$ is holomorphic on the inside of $\Gamma = \gamma_{s_2} - \gamma_{s_1}$, hence by the homology form of Cauchy's theorem $0 = \int_\Gamma f(z)/(z-z_0)^{n+1}dz = \int_{\gamma_{s_2}} f(z)/(z-z_0)^{n+1}dz - \int_{\gamma_{s_1}} f(z)/(z-z_0)^{n+1}dz$. □

Remark 20.16. Note that the above proof shows that the integral $\int_{\gamma_R} \frac{f(z)}{z-w}dz$ defines a holomorphic function of w in $B(z_0,R)$, while $\int_{\gamma_r} \frac{f(z)}{z-w}dz$ defines a holomorphic function of w on $\mathbb{C}\backslash B(z_0,r)$. Thus we have actually expressed $f(w)$ on A as the difference of two functions which are holomorphic on $B(z_0,R)$ and $\mathbb{C}\backslash\bar{B}(z_0,r)$ respectively.

Definition 20.17. Let $f\colon U\backslash S \to \mathbb{C}$ be a function which is holomorphic on a domain U except at a discrete set $S \subseteq U$. Then for any $a \in S$ the previous theorem shows that for $r > 0$ sufficiently small, we have

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n, \quad \forall z \in B(a,r)\backslash\{a\}.$$

We define

$$P_a(f) = \sum_{n=-1}^{-\infty} c_n(z-a)^n,$$

to be the *principal part* of f at a. This generalizes the previous definition we gave for the principal part of a meromorphic function. Note that the proof of Theorem 20.15 shows that the series $P_a(f)$ is uniformly convergent on $\mathbb{C}\backslash B(a,r)$ for all $r > 0$, and hence defines a holomorphic function on $\mathbb{C}\backslash\{a\}$.