Metric spaces and complex analysis
Mathematical Institute, University of Oxford
Michaelmas Term 2019

Problem Sheet 7

1. Show that the polynomial $p(z) = z^8 + 3z^3 + 7z + 5$ has two zeros in the first quadrant.
Hint: Calculate the change in argument of $p(\gamma_R(t))$ around a quarter-circle contour γ_R of radius R. For paths which are not closed, it may be useful to consider the quantity $A(\gamma, f) = \Re((2\pi i)^{-1} \int_\gamma f'(z)/f(z)dz)$, for a function f not vanishing on γ.

2. Prove, for $a > 0$, that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + a^4} = \frac{\pi}{a^3\sqrt{2}}.$$

3. Show that

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{2x - 1}\, \mathrm{d}x = -\frac{\pi}{2}.$$

4. By considering the integral

$$\int_{\Gamma_n} \frac{\pi \mathrm{d}w}{w^2 \sin \pi w}$$

where Γ_n is the square in $\mathbb{C}$ with vertices $\pm(n + 1/2)(1 \pm i)$ show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

(You may assume that there exists C such that $|\csc \pi w| \leqslant C$ on Γ_n for all n and all w.)

5. Write down a definition of a branch of $\log(z + i)$ which is holomorphic in the cut-plane

$$\mathbb{C}\backslash\{z : \operatorname{Re} z = 0,\ \operatorname{Im} z \leqslant -1\}.$$

By integrating $\log(z + i)/(z^2 + 1)$ around a suitable closed path, evaluate

$$\int_{-\infty}^{\infty} \frac{\log(x + i)}{x^2 + 1}\, \mathrm{d}x$$

and, by taking real parts, show that

$$\int_{-\infty}^{\infty} \frac{\log(x^2 + 1)}{x^2 + 1}\, \mathrm{d}x = 2\pi \log 2.$$

6. Show that

$$\int_{0}^{\infty} \frac{\sin px \sin qx}{x^2}\, \mathrm{d}x = \frac{\pi \min(p, q)}{2},$$

where $p, q > 0$.

7. Let $a \in \mathbb{C}$ with $-1 < \operatorname{Re} a < 1$. By considering a rectangular contour with corners at R, $R + i\pi$, $-R + i\pi$, $-R$, show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\, \mathrm{d}x = \pi \sec\left(\frac{\pi a}{2}\right)$$

and hence evaluate, for real n,

$$\int_{-\infty}^{\infty} \frac{\cos nx}{\cosh x}\, \mathrm{d}x.$$