Lemma 21.1. *Suppose that $f : U \to \mathbb{C}$ is a meromorphic and has a zero of order k or a pole of order k at $z_0 \in U$. Then $f'(z)/f(z)$ has a simple pole at z_0 with residue k or $-k$ respectively.*

Proof. If $f(z)$ has a zero of order k we have $f(z) = (z - z_0)^k g(z)$ where $g(z)$ is holomorphic near z_0 and $g(z_0) \neq 0$. It follows that

$$f'(z)/f(z) = \frac{k}{z - z_0} + g'(z)/g(z),$$

and since $g(z) \neq 0$ near z_0 it follows $g'(z)/g(z)$ is holomorphic near z_0, so that the result follows. The case where f has a pole at z_0 is similar. □

Remark 21.2. Note that if U is an open set on which one can define a holomorphic branch L of [Log(z)] then $g(z) = L(f(z))$ has $g'(z) = f'(z)/f(z)$. Thus integrating $f'(z)/f(z)$ along a path γ will measure the change in argument around the origin of the path $f(\gamma(t))$. The residue theorem allows us to relate this to the number of zeros and poles of f inside γ, as the next theorem shows:

Theorem 21.3. *(Argument principle): Suppose that U is an open set and $f : U \to \mathbb{C}$ is a meromorphic function on U. If $B(a,r) \subseteq U$ and N is the number of zeros (counted with multiplicity) and P is the number of poles (again counted with multiplicity) of f inside $B(a,r)$ and f has neither on $\partial B(a,r)$ then*

$$N - P = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz,$$

where $\gamma(t) = a + re^{2\pi i t}$ is a path with image $\partial B(a,r)$. Moreover this is the winding number of the path $\Gamma = f \circ \gamma$ about the origin.

Proof. It is easy to check that $I(\gamma, z)$ is 1 if $|z - a| \leq 1$ and is 0 otherwise. Since Lemma 21.1 shows that $f'(z)/f(z)$ has simple poles at the zeros and poles of f with residues the corresponding orders the result immediately from Theorem 22.1.

For the last part, note that the winding number of $\Gamma(t) = f(\gamma(t))$ about zero is just

$$\int_{f\circ\gamma} dw/w = \int_0^1 \frac{1}{f(\gamma(t))}f'(\gamma(t))\gamma'(t)dt = \int_\gamma \frac{f'(z)}{f(z)}dz$$

□

Remark 21.4. The argument principle also holds, with the same proof, to any closed path γ on which f is continuous and non-vanishing, provided it has winding number +1 around its inside. Thus for example it applies to triangles, or paths built from an arc of a circle and the line segments joining the end-points to the centre of the circle, provided they are correctly oriented.

The argument principle is very useful – we use it here to establish some important results.

Theorem 21.5. *(Rouché's theorem): Suppose that f and g are holomorphic functions on an open set U in $\mathbb{C}$ and $\bar{B}(a,r) \subset U$. If $|f(z)| > |g(z)|$ for all $z \in \partial B(a,r)$ then f and $f + g$ have the same change in argument around γ, and hence the same number of zeros in $B(a,r)$ (counted with multiplicities).*

Proof. Let $\gamma(t) = a + re^{2\pi i t}$ be a parametrization of the boundary circle of $B(a,r)$. We need to show that $(f + g)/f = 1 + g/f$ has the same number of zeros as poles (Note that $f(z) \neq 0$ on $\partial B(a,r)$ since $|f(z)| > |g(z)|$.) But by the argument principle, this number is the winding number of $\Gamma(t) = h(\gamma(t))$ about zero, where $h(z) = 1 + g(z)/f(z)$. Since, by assumption, for $z \in \gamma^*$ we have $|g(z)| < |f(z)|$ and so $|g(z)/f(z)| < 1$, the image of Γ lies entirely in $B(1,1)$ and thus in the half-plane $\{z : \Re(z) > 0\}$. Hence picking a branch of Log defined on this half-plane, we see that the integral

$$\int_\Gamma \frac{dz}{z} = \mathrm{Log}(h(\gamma(1)) - \mathrm{Log}(h(\gamma(0)) = 0$$

as required.

$\square$

Remark 21.6. Rouche's theorem can be useful in counting the number of zeros of a function f – one tries to find an approximation to f whose zeros are easier to count and then by Rouche's theorem obtain information about the zeros of f. Just as for the argument principle above, it also holds for closed paths which having winding number about their inside.

Example 21.7. Suppose that $P(z) = z^4 + 5z + 2$. Then on the circle $|z| = 2$, we have $|z|^4 = 16 > 5.2 + 2 \geq |5z + 2|$, so that if $g(z) = 5z + 2$ we see that $P - g = z^4$ and P have the same number of roots in $B(0,2)$. It follows by Rouche's theorem that the four roots of $P(z)$ all have modulus less than 2. On the other hand, if we take $|z| = 1$, then $|5z + 2| \geq 5 - 2 = 3 > |z^4| = 1$, hence $P(z)$ and $5z + 2$ have the same number of roots in $B(0,1)$. It follows $P(z)$ has one root of modulus less than 1, and 3 of modulus between 1 and 2.

Theorem 21.8. *(Open mapping theorem): Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and non-constant on a domain U. Then for any open set $V \subset U$ the set $f(V)$ is also open.*

Proof. Suppose that $w_0 \in f(V)$, say $f(z_0) = w_0$. Then $g(z) = f(z) - w_0$ has a zero at z_0 which, since f is nonconstant, is isolated. Thus we may find an $r > 0$ such that $g(z) \neq 0$ on $\bar{B}(z_0,r)\backslash\{z_0\} \subset U$ and in particular since $\partial B(z_0,r)$ is compact, we have $|g(z)| \geq \delta > 0$ on $\partial B(z_0,r)$. But then if $|w - w_0| < \delta$ it follows $|w - w_0| < |g(z)|$ on $\partial B(z_0,r)$, hence by Rouche's theorem, since $g(z)$ has a zero in $B(z_0,r)$ it follows $h(z) = g(z) + (w_0 - w) = f(z) - w$ does also, that is, $f(z)$ takes the value w in $B(z_0,r)$. Thus $B(w_0,\delta) \subseteq f(B(z_0,r))$ and hence $f(U)$ is open as required. $\square$

Remark 21.9. Note that the proof actually establishes a bit more than the statement of the theorem: if $w_0 = f(z_0)$ then the multiplicity d of the zero of the function $f(z) - w_0$ at z_0 is called the *degree* of f at z_0. The proof shows that locally the function f is d-to-1, counting multiplicities, that is, there are $r,\epsilon \in \mathbb{R}_{>0}$ such that for every $w \in B(w_0,\epsilon)$ the equation $f(z) = w$ has d solutions counted with multiplicity in the disk $B(z_0,r)$.

Theorem 21.10. *(Inverse function theorem): Suppose that $f\colon U \to \mathbb{C}$ is injective and holomorphic and that $f'(z) \neq 0$ for all $z \in U$. If $g\colon f(U) \to U$ is the inverse of f, then g is holomorphic with $g'(w) = 1/f'(g(w))$.*

Proof. By the open mapping theorem, the function g is continuous, indeed if V is open in $f(U)$ then $g^{-1}(V) = f(V)$ is open by that theorem. To see that g is holomorphic, fix $w_0 \in f(U)$ and let $z_0 = g(w_0)$. Note that since g and f are continuous, if $w \to w_0$ then $f(w) \to z_0$. Writing $z = f(w)$ we have

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = 1/f'(z_0)$$

as required. $\square$

Remark 21.11. Note that the non-trivial part of the proof of the above theorem is the fact that g is continuous! In fact the condition that $f'(z) \neq 0$ follows from the fact that f is bijective – this can be seen using the degree of f: if $f'(z_0) = 0$ and f is nonconstant, we must have $f(z) - f(z_0) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$ and $k \geq 1$. Since we can chose a holomorphic branch of $g^{1/k}$ near z_0 it follows that $f(z)$ is locally k-to-1 near z_0, which contradicts the injectivity of f. For details see the Appendices. Notice that this is in contrast with the case of a single real variable, as the example $f(x) = x^3$ shows. Once again, complex analysis is "nicer" than real analysis!

22. THE RESIDUE THEOREM

We can now prove one of the most useful theorems of the course – it is extremely powerful as a method for computing integrals, as you will see this course and many others.

Theorem 22.1. *(Residue theorem): Suppose that U is an open set in $\mathbb{C}$ and γ is a path whose inside is contained in U, so that for all $z \notin U$ we have $I(\gamma, z) = 0$. Then if $S \subset U$ is a finite set such that $S \cap \gamma^* = \emptyset$ and f is a holomorphic function on $U \backslash S$ we have*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{a \in S} I(\gamma, a)Res_a(f)$$

Proof. For each $a \in S$ let $P_a(f)(z) = \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n$ be the principal part of f at a, a holomorphic function on $\mathbb{C}\backslash\{a\}$. Then by definition of $P_a(f)$, the difference $f - P_a(f)$ is holomorphic at $a \in S$, and thus $g(z) = f(z) - \sum_{a \in S} P_a(f)$ is holomorphic on all of U. But then by Theorem 20.11 we see that $\int_\gamma g(z)dz = 0$, so that

$$\int_\gamma f(z)dz = \sum_{a \in S}\int_\gamma P_a(f)(z)dz$$

But by the proof of Theorem 20.17, the series $P_a(f)$ converges uniformly on γ^* so that

$$\begin{aligned}\int_\gamma P_a(f)dz &= \int_\gamma \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n = \sum_{n=1}^{\infty}\int_\gamma \frac{c_{-n}(a)dz}{(z-a)^n} \\ &= \int_\gamma \frac{c_{-1}(a)dz}{z-a} = I(\gamma, a)\mathrm{Res}_a(f),\end{aligned}$$

since for $n > 1$ the function $(z-a)^{-n}$ has a primitive on $\mathbb{C}\backslash\{a\}$. The result follows. □

Remark 22.2. In practice, in applications of the residue theorem, the winding numbers $I(\gamma, a)$ will be simple to compute in terms of the argument of $(z-a)$ – in fact most often they will be 0 or ± 1 as we will usually apply the theorem to integrals around simple closed curves.

22.1. **Residue Calculus.** The Residue theorem gives us a very powerful technique for computing many kinds of integrals. In this section we give a number of examples of its application.

Example 22.3. Consider the integral $\int_0^{2\pi} \frac{dt}{1+3\cos^2(t)}$. If we let γ be the path $t \mapsto e^{it}$ and let $z = e^{it}$ then $\cos(t) = \Re(z) = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(z + 1/z)$. Thus we have

$$\frac{1}{1+3\cos^2(t)} = \frac{1}{1+3/4(z+1/z)^2} = \frac{1}{1+\frac{3}{4}z^2+\frac{3}{2}+\frac{3}{4}z^{-2}} = \frac{4z^2}{3+10z^2+3z^4},$$

Finally, since $dz = izdt$ it follows

$$\int_0^{2\pi}\frac{dt}{1+3\cos^2(t)} = \int_\gamma \frac{-4iz}{3+10z^2+3z^4}dz.$$

Thus we have turned our real integral into a contour integral, and to evaluate the contour integral we just need to calculate the residues of the meromorphic function $g(z) = \frac{-4iz}{3+10z^2+3z^4}$ at the poles it has inside the unit circle. Now the poles of $g(z)$ are the zeros of the polynomial $p(z) = 3 + 10z^2 + 3z^4$, which are at $z^2 \in \{-3, -1/3\}$. Thus the poles inside the unit circle are at $\pm i/\sqrt{3}$. In particular, since p has degree 4 and has four roots, they must all be simple zeros, and so g has simple poles at these points. The residue at a simple pole z_0 can be calculated as the limit $\lim_{z \to z_0}(z - z_0)g(z)$, thus we see (compare with Remark 19.14) that

$$\begin{aligned}\mathrm{Res}_{z=\pm i/\sqrt{3}}(g(z)) &= \lim_{z \to \pm i/\sqrt{3}} \frac{-4iz(z - \pm i/\sqrt{3})}{3+10z^2+3z^4} = (\pm 4/\sqrt{3}).\frac{1}{p'(\pm i/\sqrt{3})} \\ &= (\pm 4/\sqrt{3}).\frac{1}{20(\pm i/\sqrt{3}) + 12(\pm i/\sqrt{3})^3} = 1/4i.\end{aligned}$$

It now follows from the Residue theorem that

$$\int_0^{2\pi}\frac{dt}{1+3\cos^2(t)} = 2\pi i\big(\mathrm{Res}_{z=i/\sqrt{3}}((g(z)) + \mathrm{Res}_{z=-i/\sqrt{3}}(g(z))\big) = \pi.$$

Remark 22.4. Often we are interested in integrating along a path which is not closed or even finite, for example, we might wish to understand the integral of a function on the positive real axis. The residue theorem can still be a power tool in calculating these integrals, provided we complete the path to a closed one in such a way that we can control the extra contribution to the integral along the part of the path we add.

Example 22.5. If we have a function f which we wish to integrate over the whole real line (so we have to treat it as an improper Riemann integral) then we may consider the contours Γ_R given as the concatenation of the paths $\gamma_1\colon [-R,R] \to \mathbb{C}$ and $\gamma_2\colon [0,1] \to \mathbb{C}$ where

$$\gamma_1(t) = -R + t; \quad \gamma_2(t) = Re^{i\pi t}.$$

(so that $\Gamma_R = \gamma_2 \star \gamma_1$ traces out the boundary of a half-disk). In many cases one can show that $\int_{\gamma_2} f(z)dz$ tends to 0 as $R \to \infty$, and by calculating the residues inside the contours Γ_R deduce the integral of f on $(-\infty,\infty)$. To see this strategy in action, consider the integral

$$\int_0^\infty \frac{dx}{1+x^2+x^4}.$$

It is easy to check that this integral exists as an improper Riemann integral, and since the integrand is even, it is equal to

$$\frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{1+x^2+x^4}dx.$$

If $f(z) = 1/(1+z^2+z^4)$, then $\int_{\Gamma_R} f(z)dz$ is equal to $2\pi i$ times the sum of the residues inside the path Γ_R. The function $f(z) = 1/(1+z^2+z^4)$ has poles at $z^2 = \pm e^{2\pi i/3}$ and hence at $\{e^{\pi i/3}, e^{2\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}\}$. They are all simple poles and of these only $\{\omega, \omega^2\}$ are in the upper-half plane, where $\omega = e^{i\pi/3}$. Thus by the residue theorem, for all $R > 1$ we have

$$\int_{\Gamma_R} f(z)dz = 2\pi i\big(\mathrm{Res}_\omega(f(z)) + \mathrm{Res}_{\omega^2}(f(z))\big),$$

and we may calculate the residues using the limit formula as above (and the fact that it evaluates to the reciprocal of the derivative of $1+z^2+z^4$): Indeed since $\omega^3 = -1$ we have $\mathrm{Res}_\omega(f(z)) = \frac{1}{2\omega+4\omega^3} = \frac{1}{2\omega-4}$, while $\mathrm{Res}_{\omega^2}(f(z)) = \frac{1}{2\omega^2+4\omega^6} = \frac{1}{4+2\omega^2}$. Thus we obtain:

$$\begin{aligned}
\int_{\Gamma_R} f(z)dz &= 2\pi i\Big(\frac{1}{2\omega-4} + \frac{1}{2\omega^2+4}\Big) \\
&= \pi i\Big(\frac{1}{\omega-2} + \frac{1}{\omega^2+2}\Big) \\
&= \pi i\Big(\frac{\omega^2+\omega}{2(\omega-\omega^2)-5}\Big) = -\sqrt{3}\pi/(-3) = \pi/\sqrt{3},
\end{aligned}$$

(where we used the fact that $\omega^2+\omega = i\sqrt{3}$ and $\omega - \omega^2 = 1$). Now clearly

$$\int_{\Gamma_R} f(z)dz = \int_{-R}^{R} \frac{dt}{1+t^2+t^4} + \int_{\gamma_2} f(z)dz,$$

and by the estimation lemma we have

$$\Big|\int_{\gamma_2} f(z)dz\Big| \le \sup_{z\in\gamma_2^*}|f(z)|.\ell(\gamma_2) \le \frac{\pi R}{R^4-R^2-1} \to 0,$$

as $R \to \infty$, it follows that

$$\pi/\sqrt{3} = \lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = \int_{-\infty}^{\infty} \frac{dt}{1+t^2+t^4}.$$

22.2. **Jordan's Lemma and applications.** The following lemma is a real-variable fact which is fundamental to something known as *convexity*. Note that if x, y are vectors in any vector space then the set $\{tx + (1-t)y : t \in [0,1]\}$ describes the line segment between x and y.

Lemma 22.6. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then if $[a,b]$ is an interval on which $g''(x) < 0$, the function g is convex on $[a,b]$, that is, for $x < y \in [a,b]$ we have*

$$g(tx + (1-t)y) \geq tg(x) + (1-t)g(y), \quad t \in [0,1].$$

Thus informally speaking, chords between points on the graph of g lie below the graph itself.

Proof. Given $x, y \in [a,b]$ and $t \in [0,1]$ let $\xi = tx + (1-t)y$, a point in the interval between x and y. Now the slope of the chord between $(x, g(x))$ and $(\xi, g(\xi))$ is, by the Mean Value Theorem, equal to $g'(s_1)$ where s_1 lies between x and ξ, while the slope of the chord between $(\xi, g(\xi))$ and $(y, g(y))$ is equal to $g'(s_2)$ for s_2 between ξ and y. If $g(\xi) < tg(x) + (1-t)g(y)$ it follows that $g'(s_1) < 0$ and $g'(s_2) > 0$. Thus by the mean value theorem for $g'(x)$ applied to the points s_1 and s_2 it follows there is an $s \in (s_1, s_2)$ with $g''(s) = (g'(s_2) - g'(s_1))/(s_2 - s_1) > 0$, contradicting the assumption that $g''(x)$ is negative on (a,b). □

The following lemma is an easy application of this convexity result.

Lemma 22.7. *(Jordan's Lemma): Let $f\colon \mathbb{H} \to \mathbb{C}_\infty$ be a meromorphic function on the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. Suppose that $f(z) \to 0$ as $z \to \infty$ in $\mathbb{H}$. Then if $\gamma_R(t) = Re^{it}$ for $t \in [0,\pi]$ we have*

$$\int_{\gamma_R} f(z)e^{i\alpha z} dz \to 0$$

as $R \to \infty$ for all $\alpha \in \mathbb{R}_{>0}$.

Proof. Suppose that $\epsilon > 0$ is given. Then by assumption we may find an S such that for $|z| > S$ we have $|f(z)| < \epsilon$. Thus if $R > S$ and $z = \gamma_R(t)$, it follows that

$$|f(z)e^{i\alpha z}| = \leq \epsilon e^{-\alpha R \sin(t)}.$$

But now applying Lemma 22.6 to the function $g(t) = \sin(t)$ with $x = 0$ and $y = \pi/2$ we see that $\sin(t) \geq \frac{2}{\pi}t$ for $t \in [0, \pi/2]$. Similarly we have $\sin(\pi - t) \geq 2(\pi - t)/\pi$ for $t \in [\pi/2, \pi]$. Thus we have

$$|f(z)e^{i\alpha z}| \leq \begin{cases} \epsilon.e^{-2\alpha R t/\pi}, & t \in [0, \pi/2] \\ \epsilon.e^{-2\alpha R(\pi - t)/\pi} & t \in [\pi/2, \pi] \end{cases}$$

But then it follows that

$$|\int_{\gamma_R} f(z)e^{i\alpha z} dz| \leq 2\int_0^{\pi/2} \epsilon R.e^{-2\alpha R t/\pi} dt = \epsilon.\pi \frac{1 - e^{-\alpha R}}{\alpha} < \epsilon.\pi/\alpha,$$

Thus since $\pi/\alpha > 0$ is independent of R, it follows that $\int_{\gamma_R} f(z)e^{i\alpha z} dz \to 0$ as $R \to \infty$ as required. □

Remark 22.8. If η_R is an arc of a semicircle in the upper half plane, say $\eta_R(t) = Re^{it}$ for $0 \leq t \leq 2\pi/3$, then the same proof shows that $\int_{\eta_R} f(z)e^{i\alpha z} dz$ tends to zero as R tends to infinity. This is sometimes useful when integrating around the boudary of a sector of disk (that is a set of the form $\{re^{i\theta} : 0 \leq r \leq R, \theta \in [\theta_1, \theta_2]\}$).

It is also useful to note that if $\alpha < 0$ then the integral of $f(z)e^{i\alpha z}$ around a semicircle in the *lower* half plane tends to zero as the radius of the semicircle tends to infinity provided $|f(z)| \to 0$ as $|z| \to \infty$ in the lower half plane. This follows immediately from the above applied to $f(-z)$.

Example 22.9. Consider the integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$. This is an improper integral of an even function, thus it exists if and only if the limit of $\int_{-R}^{R} \frac{\sin(x)}{x} dx$ exists as $R \to \infty$. To compute this consider the integral along the closed curve η_R given by the concatenation $\eta_R = \nu_R \star \gamma_R$, where $\nu_R\colon [-R, R] \to \mathbb{R}$ given by $\nu_R(t) = t$ and $\gamma_R(t) = Re^{it}$ (where $t \in [0, \pi]$). Now if we let $f(z) = \frac{e^{iz}-1}{z}$, then f has a removable singularity at $z = 0$

(as is easily seen by considering the power series expansion of e^{iz}) and so is an entire function. Thus we have $\int_{\eta_R} f(z)dz = 0$ for all $R > 0$. Thus we have

$$0 = \int_{\eta_R} f(z)dz = \int_{-R}^{R} f(t)dt + \int_{\gamma_R} \frac{e^{iz}}{z}dz - \int_{\gamma_R} \frac{dz}{z}.$$

Now Jordan's lemma ensures that the second term on the right tends to zero as $R \to \infty$, while the third term integrates to $\int_0^\pi \frac{iRe^{it}}{Re^{it}}dt = i\pi$. It follows that $\int_{-R}^{R} f(t)dt$ tends to $i\pi$ as $R \to \infty$. and hence taking imaginary parts we conclude the improper integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx$ is equal to π.

Remark 22.10. The function $f(z) = \frac{e^{iz}-1}{z}$ might not have been the first meromorphic function one could have thought of when presented with the previous improper integral. A more natural candidate might have been $g(z) = \frac{e^{iz}}{z}$. There is an obvious problem with this choice however, which is that it has a pole on the contour we wish to integrate around. In the case where the pole is simple (as it is for e^{iz}/z) there is standard procedure for modifying the contour: one indents it by a small circular arc around the pole. Explicitly, we replace the ν_R with $\nu_R^- \star \gamma_\epsilon \star \nu_R^+$ where $\nu_R^{\pm}(t) = t$ and $t \in [-R, -\epsilon]$ for ν_R^-, and $t \in [\epsilon, R]$ for ν_R^+ (and as above $\gamma_\epsilon(t) = \epsilon e^{i(\pi-t)}$ for $t \in [0,\pi]$). Since $\frac{\sin(x)}{x}$ is bounded at $x = 0$ the sum

$$\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}dx + \int_{\epsilon}^{R} \frac{\sin(x)}{x}dx \to \int_{-R}^{R} \frac{\sin(x)}{x}dx,$$

as $\epsilon \to 0$, while the integral along γ_ϵ can be computed explicitly: by the Taylor expansion of e^{iz} we see that $\text{Res}_{z=0}\frac{e^{iz}}{z} = 1$, so that $e^{iz} - 1/z$ is bounded near 0. It follows that as $\epsilon \to 0$ we have $\int_{\gamma_\epsilon}(e^{iz}/z - 1/z)dz \to 0$. On the other hand $\int_{\gamma_\epsilon} dz/z = \int_{-\pi}^{0}(-\epsilon i e^{i(\pi-t)})/(e^{i(\pi-t)}dt = -i\pi$, so that we see

$$\int_{\gamma_\epsilon} \frac{e^{iz}}{z}dz \to -i\pi$$

as $\epsilon \to 0$.

Combining all of this we conclude that if $\Gamma_\epsilon = \nu_R^- \star \gamma_\epsilon \star \nu_R^+ \star \gamma_R$ then

$$\begin{aligned}
0 = \int_{\Gamma_\epsilon} f(z)dz &= \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx + \int_{\gamma_\epsilon} \frac{e^{iz}}{z}dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x}dx + \int_{\gamma_R} \frac{e^{iz}}{z}dz. \\
&= 2i\int_{\epsilon}^{R} \frac{\sin(x)}{x} + \int_{\gamma_\epsilon} \frac{e^{iz}}{z} + \int_{\gamma_R} \frac{e^{iz}}{z}dz \\
&\to 2i\int_{0}^{R} \frac{\sin(x)}{x}dx - i\pi + \int_{\gamma_R} \frac{e^{iz}}{z}dz.
\end{aligned}$$

as $\epsilon \to 0$. Then letting $R \to \infty$, it follows from Jordans Lemma that the third term tends to zero so we see that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx = 2\int_{0}^{\infty} \frac{\sin(x)}{x}dx = \pi$$

as required.

We record a general version of the calculation we made for the contribution of the indentation to a contour in the following Lemma.

Lemma 22.11. *Let $f\colon U \to \mathbb{C}$ be a meromorphic function with a simple pole at $a \in U$ and let $\gamma_\epsilon\colon [\alpha, \beta] \to \mathbb{C}$ be the path $\gamma_\epsilon(t) = a + \epsilon e^{it}$, then*

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)dz = Res_a(f).(\beta - \alpha)i.$$

Proof. Since f has a simple pole at a, we may write

$$f(z) = \frac{c}{z-a} + g(z)$$

where $g(z)$ is holomorphic near z and $c = \mathrm{Res}_a(f)$ (indeed $c/(z-a)$ is just the principal part of f at a). But now as g is holomorphic at a, it is continuous at a, and so bounded. Let $M, r > 0$ be such that $|g(z)| < M$ for all $z \in B(a,r)$. Then if $0 < \epsilon < r$ we have

$$|\int_{\gamma_\epsilon} g(z)dz| \leq \ell(\gamma_\epsilon)M = (\beta - \alpha)\epsilon.M,$$

which clearly tends to zero as $\epsilon \to 0$. On the other hand, we have

$$\int_{\gamma_\epsilon} \frac{c}{z-a}dz = \int_\alpha^\beta \frac{c}{\epsilon e^{it}} i\epsilon e^{it} dt = \int_\alpha^\beta (ic)dt = ic(\beta - \alpha).$$

Since $\int_{\gamma_\epsilon} f(z)dz = \int_{\gamma_\epsilon} c/(z-a)dz + \int_{\gamma_\epsilon} g(z)dz$ the result follows. □

22.3. **On the computation of residues and principal parts.** The previous examples will hopefully have convinced you of the power of the residue theorem. Of course for it to be useful one needs to be able to calculate the residues of functions with isolated singularities. In practice the integral formulas we have obtained for the residue are often not the best way to do this. In this section we discuss a more direct approach which is often useful when one wishes to calculate the residue of a function which is given as the ratio of two holomorphic functions.

More precisely, suppose that we have a function $F\colon U \to \mathbb{C}$ given to us as a ratio f/g of two holomorphic functions f, g on U where g is non-constant. The singularities of the function F are therefore poles which are located precisely at the (isolated) zeros of the function g, so that F is meromorphic. For convenience, we assume that we have translated the plane so as to ensure the pole of F we are interested in is at $a = 0$. Let $g(z) = \sum_{n\geq 0} c_n z^n$ be the power series for g, which will converge to $g(z)$ on any $B(0,r)$ such that $\bar{B}(0,r) \subseteq U$. Since $g(0) = 0$, and this zero is isolated, there is a $k > 0$ minimal with $c_k \neq 0$, and hence

$$g(z) = c_k z^k (1 + \sum_{n\geq 1} a_n z^n),$$

where $a_n = c_{n+k}/c_k$. Now if we let $h(z) = \sum_{n=1}^\infty a_n z^{n-1}$ then $h(z)$ is holomorphic in $B(0,r)$ – since $h(z) = (g(z) - c_k z^k)/(c_k z^{k+1})$ – and moreover

$$\frac{1}{g(z)} = \frac{1}{c_k z^k}\big(1 + zh(z)\big)^{-1},$$

Now as h is continuous, it is bounded on $\bar{B}(0,r)$, say $|h(z)| < M$ for all $z \in \bar{B}(0,r)$. But then we have, for $|z| \leq \delta = \min\{r, 1/(2M)\}$,

$$\frac{1}{g(z)} = \frac{1}{c_k z^k}\big(\sum_{n=0}^\infty (-1)^n z^n h(z)^n\big),$$

where by the Weierstrass M-test, the above series converges uniformly on $\bar{B}(0,\delta)$. Moreover, for any n, the series $\sum_{m\geq n}(-1)^m z^m h(z)^m$ is a holomorphic function which vanishes to order at least n at $z = 0$, so that $\frac{1}{c_k z^k}\sum_{n\geq k}(-1)^n z^n h(z)^n$ is holmorphic. It follows that the principal part of the Laurent series of $1/g(z)$ is equal to the principal part of the function

$$\frac{1}{c_k z^k}\sum_{n=1}^k (-1)^{k-1} z^k h(z)^k.$$

Since we know the power series for $h(z)$, this allows us to compute the principal part of $\frac{1}{g(z)}$ as claimed. Finally, the principal part $P_0(F)$ of $F = f/g$ at $z = 0$ is just the $P_0(f.P_0(g))$, the principal part of the function $f(z).P_0(g)$, which again is straight-forward to compute if we know the power series expansion of

$f(z)$ at 0 (indeed we only need the first k terms of it). The best way to digest this analysis is by means of examples. We consider one next, and will examine another in the next section on summation of series.

Example 22.12. Consider $f(z) = 1/(z^2\sinh(z)^3)$. Now $\sinh(z) = (e^z - e^{-z})/2$ vanishes on $\pi i\mathbb{Z}$, and these zeros are all simple since $\frac{d}{dz}(\sinh(z)) = \cosh(z)$ has $\cosh(n\pi i) = (-1)^n \neq 0$. Thus $f(z)$ has a pole or order 5 at zero, and poles of order 3 at $\pi i n$ for each $n \in \mathbb{Z}\backslash\{0\}$. Let us calculate the principal part of f at $z = 0$ using the above technique. We will write $O(z^k)$ for the vector space of holomorphic functions which vanish to order k at 0.

$$\begin{aligned}
z^2\sinh(z)^3 &= z^2(z + \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7))^3 = z^5(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6))^3 \\
&= z^5(1 + \frac{3z^2}{3!} + \frac{3z^4}{(3!)^2} + \frac{3z^4}{5!} + O(z^6)) \\
&= z^5(1 + \frac{z^2}{2} + \frac{13z^4}{120} + O(z^6)) \\
&= z^5\left(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\right)
\end{aligned}$$

Thus, in the notation of the above discussion, $h(z) = \frac{z}{2} + \frac{13z^3}{120} + O(z^5)$, and so, as h vanishes to first order at $z = 0$, in order to obtain the principal part we just need to consider the first two terms in the geometric series $(1 + zh(z))^{-1} = \sum_{n=0}^{\infty}(-1)^n z^n h(z)^n$:

$$\begin{aligned}
1/z^2\sinh(z)^3 &= z^{-5}\big(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\big)^{-1} \\
&= z^{-5}\big(1 - z(\frac{z}{2} + \frac{13z^3}{120}) + z^2\frac{z^2}{(2!)^2} + O(z^5)\big) \\
&= z^{-5}\big(1 - \frac{z^2}{2} + (\frac{1}{4} - \frac{13}{120})z^4 + O(z^5)\big) \\
&= \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z} + O(z).
\end{aligned}$$

Thus the principal part of $f(z)$ at 0 is $P_0(f) = \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z}$, and $\mathrm{Res}_0(f) = 17/120$.

There are other variants on the above method which we could have used: For example, by the binomial theorem for an arbitrary exponent we know that if $|z| < 1$ then $(1+z)^{-3} = \sum_{n\geq 0}\binom{-3}{n}z^n = 1 - 3z + 6z^2 + \ldots$. Arguing as above, it follows that for small enough z we have

$$\begin{aligned}
\sinh(z)^{-3} &= z^{-3}.(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6))^{-3} \\
&= z^{-3}\left(1 + (-3)(\frac{z^2}{3!} + \frac{z^4}{5!}) + 6(\frac{z^2}{3!} + \frac{z^4}{5!})^2 + O(z^6)\right) \\
&= z^{-3}\left(1 - \frac{z^2}{2} + (\frac{-3}{5!} + \frac{6}{(3!)^2})z^4 + O(z^6)\right) \\
&= z^{-3}\left(1 - \frac{z^2}{2} + \frac{17z^4}{120} + O(z^6)\right)
\end{aligned}$$

yielding the same result for the principal part of $1/z^2\sinh(z)^3$.

22.4. **Summation of infinite series.** Residue calculus can also be a useful tool in calculating infinite sums, as we now show. For this we use the function $f(z) = \cot(\pi z)$. Note that since $\sin(\pi z)$ vanishes precisely at the integers, $f(z)$ is meromorphic with poles at each integer $n \in \mathbb{Z}$. Moreover, since f is periodic with period 1, in order to understand the poles of f it suffices to calculate the principal part of f at $z = 0$. We can use the method of the previous section to do this:

We have $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)$, so that $\sin(z)$ vanishes with multiplicity 1 at $z = 0$ and we may write $\sin(z) = z(1 - zh(z))$ where $h(z) = z/3! - z^3/5! + O(z^5)$ is holomorphic at $z = 0$. Then

$$\frac{1}{\sin(z)} = \frac{1}{z}(1 - zh(z))^{-1} = \frac{1}{z}\big(1 + \sum_{n\geq 1} z^n h(z)^n\big) = \frac{1}{z} + h(z) + O(z^2).$$

Multiplying by $\cos(z)$ we see that the principal part of $\cot(z)$ is the same as that of $\frac{1}{z}\cos(z)$ which, using the Taylor expansion of $\cos(z)$, is clearly $\frac{1}{z}$ again. By periodicity, it follows that $\cot(\pi z)$ has a simple pole with residue $1/\pi$ at each integer $n \in \mathbb{Z}$.

We can also use this strategy[50] to find further terms of the Laurent series of $\cot(z)$: Since our $h(z)$ actually vanishes at $z = 0$, the terms $h(z)^n z^n$ vanish to order $2n$. It follows that we obtain all the terms of the Laurent series of $\cot(z)$ at 0 up to order 3, say, just by considering the first two terms of the series $1 + \sum_{n\geq 1} z^n h(z)^n$, that is, $1 + zh(z)$. Since $\cos(z) = 1 - z^2/2! + z^4/4!$, it follows that $\cot(z)$ has a Laurent series

$$\begin{aligned}\cot(z) &= (1 - \frac{z^2}{2!} + O(z^4)).\big(\frac{1}{z} + (\frac{z}{3!} - \frac{z^3}{5!} + O(z^5))\big)\\ &= \frac{1}{z} - \frac{z}{3} + O(z^3)\end{aligned}$$

The fact that $f(z)$ has simple poles at each integer will allow us to sum infinite series with the help of the following:

Lemma 22.13. *Let $f(z) = \cot(\pi z)$ and let Γ_N denotes the square path with vertices $(N+1/2)(\pm 1 \pm i)$. There is a constant C independent of N such that $|f(z)| \leq C$ for all $z \in \Gamma_N^*$.*

Proof. We need to consider the horizontal and vertical sides of the square separately. Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$. Thus on the horizontal sides of Γ_N where $z = x \pm (N+1/2)i$ and $-(N+1/2) \leq x \leq (N+1/2)$ we have

$$\begin{aligned}|\cot(\pi z)| &= \left|\frac{e^{i\pi(x\pm(N+1/2)i)} + e^{-i\pi(x\pm(N+1/2)i)}}{e^{i\pi(x\pm(N+1/2)i} - e^{-i\pi(x\pm(N+1/2)i)}}\right|\\ &\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}}\\ &= \coth(\pi(N+1/2)).\end{aligned}$$

Now since $\coth(x)$ is a decreasing function for $x \geq 0$ it follows that on the horizontal sides of Γ_N we have $|\cot(\pi z)| \leq \coth(3\pi/2)$.

On the vertical sides we have $z = \pm(N+1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$. Observing that $\cot(z + N\pi) = \cot(z)$ for any integer N and that $\cot(z + \pi/2) = -\tan(z)$, we find that if $z = \pm(N+1/2) + iy$ for any $y \in \mathbb{R}$ then

$$|\cot(\pi z)| = |-\tan(iy)| = |-\tanh(y)| \leq 1.$$

Thus we may set $C = \max\{1, \coth(3\pi/2)\}$. □

We now show how this can be used to sum an infinite series:

Example 22.14. Let $g(z) = \cot(\pi z)/z^2$. By our discussion of the poles of $\cot(\pi z)$ above it follows that $g(z)$ has simple poles with residues $\frac{1}{\pi n^2}$ at each non-zero integer n and residue $-\pi/3$ at $z = 0$.

[50] See Appendix II for more details on the generalities and justification of this method.

Consider now the integral of $g(z)$ around the paths Γ_N: By Lemma 22.13 we know $|g(z)| \leq C/|z|^2$ for $z \in \Gamma_N^*$, and for all $N \geq 1$. Thus by the estimation lemma we see that

$$\left(\int_{\Gamma_N} g(z)dz\right) \leq C.(4N+2)/(N+1/2)^2 \to 0,$$

as $N \to \infty$. But by the residue theorem we know that

$$\int_{\Gamma_N} g(z)dz = -\pi/3 + \sum_{\substack{n\neq 0,\\ -N\leq n\leq N}} \frac{1}{\pi n^2}.$$

It therefore follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

Remark 22.15. Notice that the contours Γ_N and the function $\cot(\pi z)$ clearly allows us to sum other infinite series in a similar way – for example if we wished to calculate the sum of the infinite series $\sum_{n\geq 1} \frac{1}{n^2+1}$ then we would consider the integrals of $g(z) = \cot(\pi z)/(1+z^2)$ over the contours Γ_N.

Remark 22.16. (*Non-examinable – for interest only!*): Note that taking $g(z) = (1/z^{2k})\cot(\pi z)$ for any positive integer k, the above strategy gives a method for computing $\sum_{n=1}^{\infty} 1/n^{2k}$ (check that you see why we need to take even powers of n). The analysis for the case $k = 1$ goes through in general, we just need to compute more and more of the Laurent series of $\cot(\pi z)$ the larger we take k to be.

One can show that $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ converges to a holomorphic function of s for any $s \in \mathbb{C}$ with $\Re(s) > 1$ (as usual, we define $n^s = \exp(s.\log(n))$ where log is the ordinary real logarithm). As $s \to 1$ it can be checked that $\zeta(s) \to \infty$, however it can be shown that $\zeta(s)$ extends to a meromorphic function on all of $\mathbb{C}\backslash\{1\}$. The identity theorem shows that this extension is unique if it exists[51]. (This uniqueness is known as the principle of "analytic continuation".) The location of the zeros of the ζ-function is the famous Riemann hypothesis: apart from the "trivial zeros" at negative even integers, they are conjectured to all lie on the line $\Re(z) = 1/2$. Its values at special points however are also of interest: Euler was the first to calculate $\zeta(2k)$ for positive integers k, but the values $\zeta(2k+1)$ (for k a positive integer) remain mysterious – it was only shown in 1978 by Roger Apéry that $\zeta(3)$ is irrational for example. Our analysis above is sufficient to determine $\zeta(2k)$ once one succeeds in computing explicitly the Laurent series for $\cot(\pi z)$ or equivalently the Taylor series of $z\cot(\pi z) = iz + 2iz/(e^{2iz} - 1)$. See Appendix IV for more details.

22.5. **Keyhole contours.** There are many ingenious paths which can be used to calculate integrals via residue theory. One common contour is known (for obvious reasons) as a *keyhole contour*. It is constructed from two circular paths of radius ϵ and R, where we let R become arbitrarily large, and ϵ arbitrarily small, and we join the two circles by line segments with a narrow neck in between. Explicitly, if $0 < \epsilon < R$ are given, pick a $\delta > 0$ small, and set $\eta_+(t) = t + i\delta$, $\eta_-(t) = (R-t) - i\delta$, where in each case t runs over the closed intervals with endpoints such that the endpoints of $\eta_\pm$ lie on the circles of radius ϵ and R about the origin. Let γ_R be the positively oriented path on the circle of radius R joining the endpoints of η_+ and η_- on that circle (thus traversing the "long" arc of the circle between the two points) and similarly let γ_ϵ the path on the circle of radius ϵ which is negatively oriented and joins the endpoints of $\gamma_\pm$ on the circle of radius ϵ. Then we set $\Gamma_{R,\epsilon} = \eta_+ \star \gamma_R \star \eta_- \star \gamma_\epsilon$ (see Figure 5). The keyhole contour can sometimes be useful to evaluate real integrals where the integrand is multi-valued as a function on the complex plane, as the next example shows:

Example 22.17. Consider the integral $\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$. Let $f(z) = z^{1/2}/(1+z^2)$, where we use the branch of the square root function which is continuous on $\mathbb{C}\backslash\mathbb{R}_{>0}$, that is, if $z = re^{it}$ with $t \in [0, 2\pi)$ then $z^{1/2} = r^{1/2}e^{it/2}$.

[51] It is this uniqueness and the fact that one can readily compute that $\zeta(-1) = -1/12$ that results in the rather outrageous formula $\sum_{n=1}^{\infty} n = -1/12$.

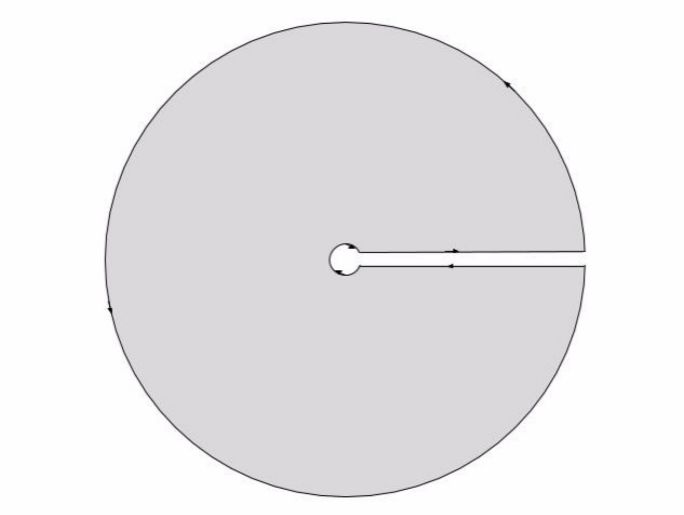

FIGURE 5. A keyhole contour.

We use the keyhole contour $\Gamma_{R,\epsilon}$. On the circle of radius R, we have $|f(z)| \leq R^{1/2}/(R^2-1)$, so by the estimation lemma, this contribution to the integral of f over $\Gamma_{R,\epsilon}$ tends to zero as $R \to \infty$. Similarly, $|f(z)|$ is bounded by $\epsilon^{1/2}/(1-\epsilon^2)$ on the circle of radius ϵ, thus again by the estimation lemma this contribution to the integral of f over $\Gamma_{R,\epsilon}$ tends to zero as $\epsilon \to 0$. Finally, the discontinuity of our branch of $z^{1/2}$ on $\mathbb{R}_{>0}$ ensures that the contributions of the two line segments of the contour do not cancel but rather both tend to $\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$ as δ and ϵ tend to zero.

To compute $\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$ we evaluate the integral $\int_{\Gamma_{R,\epsilon}} f(z)dz$ using the residue theorem: The function $f(z)$ clearly has simple poles at $z = \pm i$, and their residues are $\frac{1}{2}e^{-\pi i/4}$ and $\frac{1}{2}e^{5\pi i/4}$ respectively. It follows that

$$\int_{\Gamma_{R,\epsilon}} f(z)dz = 2\pi i\left(\frac{1}{2}e^{-\pi i/4} + \frac{1}{2}e^{5\pi i/4}\right) = \pi\sqrt{2}.$$

Taking the limit as $R \to \infty$ and $\epsilon \to 0$ we see that $2\int_0^\infty \frac{x^{1/2}}{1+x^2}dx = \pi\sqrt{2}$, so that

$$\int_0^\infty \frac{x^{1/2}dx}{1+x^2} = \frac{\pi}{\sqrt{2}}.$$