Another important feature of the stereographic projection map is that it is *conformal*, meaning that it preserves angles. The following definition helps us to formalize what this means:

Definition 23.1. If $\gamma\colon [-1,1] \to \mathbb{C}$ is a C^1 path which has $\gamma'(t) \neq 0$ for all t, then we say that the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is the *tangent line* to γ at $\gamma(t)$, and the vector $\gamma'(t)$ is a tangent vector at $\gamma(t) \in \mathbb{C}$.

Remark 23.2. Note that this definition gives us a notion of tangent vectors at points on subsets of $\mathbb{R}^n$, since the notion of a C^1 path extends readily to paths in $\mathbb{R}^n$ (we just require all n component functions are continuously differentiable). In particular, if $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$ as above, a C^1 path on $\mathbb{S}$ is simply a path $\gamma\colon [a,b] \to \mathbb{R}^3$ whose image lies in $\mathbb{S}$. It is easy to check that the tangent vectors at a point $p \in \mathbb{S}$ all lie in the plane perpendicular to p – simply differentiate the identity $f(\gamma(t)) = 1$ where $f(x,y,z) = x^2 + y^2 + z^2$ using the chain rule.

We can now state what we mean by a conformal map:

Definition 23.3. Let U be an open subset of $\mathbb{C}$ and suppose that $T\colon U \to \mathbb{C}$ (or $\mathbb{S}$) is continuously differentiable in the real sense (so all its partial derivatives exist and are continuous). If $\gamma_1, \gamma_2\colon [-1,1] \to U$ are two paths with $z_0 = \gamma_1(0) = \gamma_2(0)$ then $\gamma_1'(0)$ and $\gamma_2'(0)$ are two tangent vectors at z_0, and we may consider the angle between them (formally speaking this is the difference of their arguments). By our assumption on T, the compositions $T \circ \gamma_1$ and $T \circ \gamma_2$ are C^1-paths through $T(z_0)$, thus we obtain a pair of tangent vectors at $T(z_0)$. We say that T is *conformal* at z_0 if for every pair of C^1 paths γ_1, γ_2 through z_0, the angle between their tangent vectors at z_0 is equal to the angle between the tangent vectors at $T(z_0)$ given by the C^1 paths $T \circ \gamma_1$ and $T \circ \gamma_2$. We say that T is conformal on U if it is conformal at every $z \in U$.

One of the main reasons we focus on conformal maps here is because holomorphic functions give us a way of producing many examples of them, as the following result shows.

Proposition 23.4. *Let $f\colon U \to \mathbb{C}$ be a holomorphic map and let $z_0 \in U$ be such that $f'(z_0) \neq 0$. Then f is conformal at z_0. In particular, if $f\colon U \to \mathbb{C}$ is has nonvanishing derivative on all of U, it is conformal on all of U (and locally a biholomorphism).*

Proof. We need to show that f preserves angles at z_0. Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$. By the Chain Rule (see Lemma 26.7) we see that $\eta_1'(t) = Df_{z_0}(\gamma_1'(t))$ and $\eta_2'(t) = Df_{z_0}(\gamma_2'(t))$, and moreover if $f'(z_0) = \rho.e^{i\theta}$, then

$$Df_{z_0} = \rho.\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

(since the linear map given by multiplication by $f'(z_0)$ is precisely scaling by ρ and rotating by θ). It follows that if ϕ_1 and ϕ_2 are the arguments of $\gamma_1'(0)$ and $\gamma_2'(0)$, then the arguments of $\eta_1'(0)$ and $\eta_2'(0)$ are $\phi_1 + \theta$ and $\phi_2 + \theta$ respectively. It follows that the difference between the two pairs of arguments, that is, the angles between the curves at z_0 and $f(z_0)$, are the same.

For the final part, note that if $f'(z_0) \neq 0$ then by the definition of the degree of vanishing, the function $f(z)$ is locally biholomorphic (see the proof of the inverse function theorem). □

Example 23.5. The function $f(z) = z^2$ has $f'(z)$ nonzero everywhere except the origin. It follows f is a conformal map from $\mathbb{C}^\times$ to itself. Note that the condition that $f'(z)$ is non-zero is necessary – if we consider the function $f(z) = z^2$ at $z = 0$, $f'(z) = 2z$ which vanishes precisely at $z = 0$, and it is easy to check that at the origin f in fact doubles the angles between tangent vectors.

Lemma 23.6. *The sterographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ is conformal.*

Proof. Let z_0 be a point in $\mathbb{C}$, and suppose that $\gamma_1(t) = z_0 + tv_1$ and $\gamma_2(t) = z_0 + tv_2$ are two paths[52] having tangents v_1 and v_2 at $z_0 = \gamma_1(0) = \gamma_2(0)$. Then the lines L_1 and L_2 they describe, together with the point N, determine planes H_1 and H_2 in $\mathbb{R}^3$, and moreover the image of the lines under stereographic projection is the intersection of these planes with $\mathbb{S}$. Since the intersection of $\mathbb{S}$ with any plane is either empty or a circle, it follows that the paths γ_1 and γ_2 get sent to two circles C_1 and C_2 passing through $P = S(z_0)$ and N. Now by symmetry, these circles meet at the same angle at N as they do at P. Now the tangent lines of C_1 and C_2 at N are just the intersections of H_1 and H_2 with the plane tangent to $\mathbb{S}$ at N. But this means the angle between them will be the same as that between the intersection of H_1 and H_2 with the complex plane, since it is parallel to the tangent plane of $\mathbb{S}$ at N. Thus the angles between C_1 and C_2 at P and L_1 and L_2 at z_0 coincide as required. □

Although it follows easily from what we have already done, it is worth high-lighting the following:

Lemma 23.7. *Mobius transformations are conformal.*

Proof. As we have already shown, any holomorphic map is conformal wherever its derivative is nonzero. Since a Mobius transformation f is invertible everywhere with holomorphic inverse, its derivative must be nonzero everywhere and we are done.

One can also give a more explicit proof: If $f(z) = \frac{az+b}{cz+d}$ then it is easy to check that

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0,$$

for all $z \neq -d/c$, thus f is conformal at each $z \in \mathbb{C}\backslash\{-d/c\}$. Checking at $z = \infty, -d/c$ is similar: indeed at $\infty = [1:0]$ we use the map $i_\infty \colon \mathbb{C} \to \mathbb{P}^1$ given by $w \mapsto [1:w]$ to obtain $f_\infty(w) = \frac{a+bw}{c+dw}$ and $f'_\infty(w) = \frac{bc-ad}{(c+dw)^2}$, which is certainly nonzero at $w = 0$ (and $i_\infty(0) = \infty$). □

Since a Mobius map is given by the four entries of a 2×2 matrix, up to simultaneus rescaling, the following result is perhaps not too surprising.

Proposition 23.8. *If z_1, z_2, z_3 and w_1, w_2, w_3 are triples of pairwise distinct complex numbers, then there is a unique Mobius transformation f such that $f(z_i) = w_i$ for each $i = 1, 2, 3$.*

Proof. It is enough to show that, given any triple (z_1, z_2, z_3) of complex numbers, we can find a Mobius transformations which takes z_1, z_2, z_3 to $0, 1, \infty$ respectively. Indeed if f_1 is such a transformation, and f_2 takes $0, 1, \infty$ to w_1, w_2, w_3 respectively, then clearly $f_2 \circ f_1^{-1}$ is a Mobius transformation which takes z_i to w_i for each i.

Now consider

$$f(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

It is easy to check that $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$, and clearly f is a Mobius transformation as required. If any of z_1, z_2 or z_3 is ∞, then one can find a similar transformation (for example by letting $z_i \to \infty$ in the above formula). Indeed if $z_1 = \infty$ then we set $f(z) = \frac{z_2-z_3}{z-z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z-z_1}{z-z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z-z_1}{z_2-z_1}$.

To see the f is unique, suppose f_1 and f_2 both took z_1, z_2, z_3 to w_1, w_2, w_3. Then taking Mobius transformations g, h sending z_1, z_2, z_3 and w_1, w_2, w_3 to $0, 1, \infty$ the transformations hf_1g^{-1} and hf_2g^{-1} both take $(0, 1, \infty)$ to $(0, 1, \infty)$. But suppose $T(z) = \frac{az+b}{cz+d}$ is any Mobius transformation with $T(0) = 0$, $T(1) = 1$ and $T(\infty) = \infty$. Since T fixes ∞ it follows $c = 0$. Since $T(0) = 0$ it follows that $b/d = 0$ hence $b = 0$, thus $T(z) = a/d.z$, and since $T(1) = 1$ it follows $a/d = 1$ and hence $T(z) = z$. Thus we see that $hf_1g^{-1} = hf_2g^{-1} = \mathrm{id}$ are all the identity, and so $f_1 = f_2$ as required. □

[52]with domain $[-1, 1]$ say – or even the whole real line, except that it is non-compact.

Example 23.9. The above lemma shows that we can use Mobius transformations as a source of conformal maps. For example, suppose we wish to find a conformal transformation which takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0,1)$. The boundary of $\mathbb{H}$ is the real line, and we know Mobius transformations take lines to lines or circles, and in the latter case this means the point $\infty \in \mathbb{C}_\infty$ is sent to a finite complex number. Now any circle is uniquely determined by three points lying on it, and we know Mobius transformations allow us to take any three points to any other three points. Thus if we take f the Mobius map which sends $0 \mapsto -i$, and $1 \mapsto 1$, $\infty \mapsto i$ the real axis will be sent to the unit circle. Now we have

$$f(z) = \frac{iz+1}{z+i}$$

(one can find f in a similar fashion to the proof of Proposition 23.8).

So far, we have found a Mobius transformation which takes the real line to the unit circle. Since $\mathbb{C}\backslash\mathbb{R}$ has two connected components, the upper and lower half planes, $\mathbb{H}$ and $i\mathbb{H}$, and similarly $\mathbb{C}\backslash\mathbb{S}^1$ has two connected components, $B(0,1)$ and $\mathbb{C}\backslash\bar{B}(0,1)$. Since a Mobius transformation is continuous, it maps connected sets to connected sets, thus to check whether $f(\mathbb{H}) = B(0,1)$ it is enough to know which component of $\mathbb{C}\backslash\mathbb{S}^1$ a single point in $\mathbb{H}$ is sent to. But $f(i) = 0 \in B(0,1)$, so we must have $f(\mathbb{H}) = B(0,1)$ as required.

Note that if we had taken $g(z) = (z+i)/(iz+1)$ for example, then g also maps $\mathbb{R}$ to the unit circle $\mathbb{S}^1$, but $g(-i) = 0$, so[53] g maps the lower half plane to $B(0,1)$. If we had used this transformation, then it would be easy to "correct" it to get what we wanted: In fact there are (at least) two simple things one could do: First, one could note that the map $R(z) = -z$ (a rotation by π) sends the upper half plane to the lower half place, so that the composition $g \circ R$ is a Mobius transformation taking $\mathbb{H}$ to $B(0,1)$. Alternatively, the inversion $j(z) = 1/z$ sends $\mathbb{C}\backslash\bar{B}(0,1)$ to $B(0,1)$, so that $j \circ g$ also sends $\mathbb{H}$ to $B(0,1)$. Explicitly, we have

$$g \circ R(z) = \frac{z-i}{iz-1} = \frac{-i(iz+1)}{i(z+i)} = -f(z), \quad j \circ g(z) = \frac{iz+1}{z+i} = f(z).$$

Note in particular that f is far from unique – indeed if f is any Mobius transformation which takes $\mathbb{H}$ to $B(0,1)$ then composing it with any Mobius transformation which preserves $B(0,1)$ will give another such map. Thus for example $e^{i\theta}.f$ will be another such transformation.

Exercise 23.10. Every Mobius transformation gives a biholomorphic map from $\mathbb{C}_\infty$ to itself, but they may not preserve the distance function d_S on $\mathbb{P}^1$. What is the subgroup of Mob which are isometries of $\mathbb{P}^1$ with respect to the distance function d_S?

Given two domains D_1, D_2 in the complex plane, one can ask if there is a conformal transformation $f : D_1 \to D_2$. Since a conformal transformation is in particular a homeomorphism, this is clearly not possible for completely arbitrary domains. However if we restrict to simply-connected domains (that is, domains in which any path can be continuously deformed to any other path with the same endpoints), the following remarkable theorem shows that the answer to this question is yes! Since it will play a distinguished role later, we will write $\mathbb{D}$ for the unit disc $B(0,1)$.

Theorem 23.11. *(Riemann's mapping theorem): Let U be an open connected and simply-connected proper subset of $\mathbb{C}$. Then for any $z_0 \in U$ there is a unique bijective conformal transformation $f : U \to \mathbb{D}$ such that $f(z_0) = 0$, $f'(z_0) > 0$.*

Remark 23.12. The proof of this theorem is beyond the scope of this course, but it is a beautiful and fundamental result. The proof in fact only uses the fact that on a simply-connected domain any holomorphic function has a primitive, and hence it in fact shows that such domains are simply-connected in the topological sense (since a conformal transformation is in particular a homeomorphism, and the disc

[53]A Mobius map is a continuous function on $\mathbb{C}_\infty$, and if we remove a circle from $\mathbb{C}_\infty$ the complement is a disjoint union of two connected components, just the same as when we remove a line or a circle from the plane, thus the connectedness argument works just as well when we include the point at infinity.

in simply-connected). It relies crucially on *Montel's theorem* on families of holomorphic functions, see for example the text of Shakarchi and Stein[54] for an exposition of the argument.

Note that it follows immediately from Liouville's theorem that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0,1)$, so the whole complex plane is indeed an exception. The uniqueness statement of the theorem reduces to the question of understanding the conformal transformations of the disk $\mathbb{D}$ to itself.

Of course knowing that a conformal transformation between two domains D_1 and D_2 exists still leaves the challenge of constructing one. As we will see in the next section on harmonic maps, this is an important question. In simple cases one can often do so by hand, as we now show.

In addition to Mobius transformations, it is often useful to use the exponential function and branches of the multifunction $[z^\alpha]$ (away from the origin) when constructing conformal maps. We give an example of the kind of constructions one can do:

Example 23.13. Let $D_1 = B(0,1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so Riemann's mapping theorem ensure that there is a conformal map sending D_2 to D_1. To construct such a map, note that the domain is defined by the two curves $\gamma(0,1)$ and the real axis. It can be convenient to map the two points of intersection of these curves, ± 1 to 0 and ∞. We can readily do this with a Mobius transformation:

$$f(z) = \frac{z-1}{z+1},$$

Now since f is a Mobius transformation, it follows that $f_1(\mathbb{R})$ and $f_1(\gamma(0,1))$ are lines (since they contain ∞) passing through the origin. Indeed $f(\mathbb{R}) = \mathbb{R}$, and since f had inverse $f^{-1} = \frac{z+1}{z-1}$ it follows that the image of $\gamma(0,1)$ is $\{w \in \mathbb{C} : |w-1| = |w+1|\}$, that is, the imaginary axis. Since $f(i/2) = (-3+4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.

Now the squaring map $s\colon \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ maps Q bijectively to the half-plane $H = \{w \in \mathbb{C} : \Im(w) < 0\}$, and is conformal except at $z = 0$ (which is on the boundary, not in the interior, of Q). We may then use a Mobius map to take this half-plane to the unit disc: indeed in Example 23.9 we have already seen that the Mobius transformation $g(z) = \frac{z+i}{iz+1}$ takes the lower-half plane to the upper-half plane.

Putting everything together, we see that $F = g \circ s \circ f$ is a conformal transformation taking D_1 to D_2 as required. Calculating explicitly we find that

$$F(z) = i\left(\frac{z^2+2iz+1}{z^2-2iz+1}\right)$$

Remark 23.14. Note that there are couple of general principles one should keep in mind when constructing conformal transformations between two domains D_1 and D_2. Often if the boundary of D_1 has distinguished points (such as ± 1 in the above example) it is convenient to move these to "standard" points such as 0 and ∞, which one can do with a Mobius transformation. The fact that Mobius transformations are three-transitive and takes lines and circles to lines and circles and moreover act transitively on such means that we can always use Mobius transformations to match up those parts of the boundary of D_1 and D_2 given by line segments or arcs of circles. However these will not be sufficient in general: indeed in the above example, the fact that the boundary of D_1 is a union of a semicircle and a line segment, while that of D_2 is just a circle implies there is no Mobius transformation taking D_1 to D_2, as it would have to take ∂D_1 to ∂D_2, which would mean that its inverse would not take the unit circle to either a line or a circle. Branches of fractional power maps $[z^\alpha]$ are often useful as they allow us to change the angle at the points of intersection of arcs of the boundary (being conformal on the interior of the domain but not on its boundary).

[54] *Complex Analysis*, Princeton Lecture in Analysis II, E. M. Stein & R. Shakarchi. P.U.P.

23.1. **Conformal transformations and the Laplace equation.** In this section we will use the term *conformal map* or *conformal transformation* somewhat abusively to mean a holomorphic function whose derivative is nowhere vanishing on its domain of definition. (We have seen already that this implies the function is conformal in the sense of the previous section.) If there is a bijective conformal transformation between two domains U and V we say they are *conformally equivalent*.

Recall that a function $v\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if it is twice differentiable and $\partial_x^2 v + \partial_y^2 v = 0$. Often one seeks to find solutions to this equation on a domain $U \subset \mathbb{R}^2$ where we specify the values of v on the boundary ∂U of U. This problem is known as the *Dirichlet problem*, and makes sense in any dimension (using the appropriate Laplacian). In dimension 2, complex analysis and in particular conformal maps are a powerful tool by which one can study this problem, as the following lemma show.

Lemma 23.15. *Suppose that $U \subset \mathbb{C}$ is a simply-connected open subset of $\mathbb{C}$ and $v\colon U \to \mathbb{R}$ is twice continuously differentiable and harmonic. Then there is a holomorphic function $f\colon U \to \mathbb{C}$ such that $\Re(f) = v$. In particular, any such function v is analytic.*

Proof. (*Sketch*): Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

so that g satisfies the Cauchy-Riemann equations. It follows from Theorem 14.9 that g is holomorphic. Now since U is simply-connected, it follows that g has a primitive $G\colon U \to \mathbb{C}$. But then it follows that if $G = a(z) + ib(z)$ we have $\partial_z G = \partial_x a - i\partial_y a = g(z) = \partial_x v - i\partial_y v$, hence the partial derivatives of a and v agree on all of U. But then if $z_0, z \in U$ and γ is a path between then, the chain rule[55] shows that

$$\begin{aligned} \Re\left(\int_\gamma (\partial_x v - i\partial_y v)dz\right) &= \Re\left(\int_0^1 (\partial_x(v(\gamma(t)) - i\partial_y v(\gamma(t)))\gamma'(t)dt\right) \\ &= \int_0^1 \frac{d}{dt}(v(\gamma(t)))dt = v(z) - v(z_0), \end{aligned}$$

Similarly, we see that the same path integral is also equal to $a(z) - a(z_0)$. It follows that $a(z) = v(z) + (a(z_0) - v(z_0))$, thus if we set $f(z) = G(z) - (G(z_0) - v(z_0))$ we obtain a holomorphic function on U whose real part is equal to v as required.

Since we know that any holomorphic function is analytic, it follows that v is analytic (and in particular, infinitely differentiable). □

The previous Lemma shows that, at least locally (in a disk say) harmonic functions and holomorphic functions are in correspondence – given a holomorphic function f we obtain a harmonic function by taking its real part, while if u is harmonic the previous lemma shows we can associate to it a holomorphic function f whose real part equals u (and in fact examining the proof, we see that f is actually unique up to a purely imaginary constant). Thus if we are seeking a harmonic function on an open set U whose values are a given function g on ∂U, then it suffices to find a holomorphic function f on U such that $\Re(f) = g$ on the boundary ∂U.

Now if $H\colon U \to V$ was a bijective conformal transformation which extends to a homeomorphism $\bar{H}\colon \bar{U} \to \bar{V}$ which thus takes ∂U homeomorphically to ∂V, then if $f\colon V \to \mathbb{C}$ is holomorphic, so is $f \circ H$. Thus in particular $\Re(f \circ H)$ is a harmonic function on U. It follows that we can use conformal transformations to transport solutions of Laplace's equation from one domain to another: if we can use a conformal transformation H to take a domain U to a domain V where we already have a supply of holomorphic functions satisfying various boundary conditions, the conformal transformation H gives us a

[55]This uses the chain rule for a composition $g \circ f$ of real-differentiable functions $f\colon \mathbb{R} \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$, applied to the real and imaginary parts of the integrand. This follows in exactly the same way as the proof of Lemma 26.7. See the remark after the proof of that lemma.

corresponding set of holomorphic (and hence harmonic) functions on U. We state this a bit more formally as follow:

Lemma 23.16. *If U and V are domains and $G\colon U \to V$ is a conformal transformation, then if $u\colon V \to \mathbb{R}$ is a harmonic function on V, the composition $u \circ G$ is harmonic on U.*

Proof. To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$. If $w_0 = G(z_0)$, the continuity of G ensures we can find $\delta, \epsilon > 0$ such that $G(B(z_0,\delta)) \subseteq B(w_0,\epsilon) \subseteq V$. But now since $B(w_0,\epsilon)$ is simply-connected we know by Lemma 23.15 we can find a holomorphic function $f(z)$ with $u = \Re(f)$. But then on $B(z_0,\delta)$ we have $u \circ G = \Re(f \circ G)$, and by the chain rule $f \circ G$ is holomorphic, so that its real part is harmonic as required. □

Remark 23.17. You can also give a more direct computational proof of the above Lemma. Note also that we only need G to be holomorphic – the fact that it is a conformal equivalence is not necessary. On the other hand if we are trying to produce harmonic functions with prescribed boundary values, then we will need to use carefully chosen conformal transformations.

This strategy for studying harmonic functions might appear over-optimistic, in that the domains one can obtain from a simple open set like $B(0,1)$ or the upper-half plane $\mathbb{H}$ might consist of only a small subset of the open sets one might be interested in. However, the Riemann mapping theorem (Theorem 23.11) show that *every* domain which is simply connected, other than the whole complex plane itself, is in fact conformally equivalent to $B(0,1)$. Thus a solution to the Dirichlet problem for the disk at least in principal comes close[56] to solving the same problem for any simply-connected domain! For convenience, we will write $\mathbb{D}$ for the open disk $B(0,1)$ of radius 1 centred at 0.

In the course so far, the main examples of conformal transformations we have are the following:

(1) The exponential function is conformal everywhere, since it is its own derivative and it is everywhere nonzero.
(2) Mobius transformations understood as maps on the extended complex plane are everywhere conformal.
(3) Fractional exponents: In cut planes the functions $z \mapsto z^\alpha$ for $\alpha \in \mathbb{C}$ are conformal (the cut removes the origin, where the derivative may vanish).

Let us see how to use these transformations to obtain solutions of the Laplace equation. First notice that Cauchy's integral formula suggests a way to produce solutions to Laplace's equation in the disk: Suppose that u is a harmonic function defined on $B(0,r)$ for some $r > 1$. Then by Lemma 23.15 we know there is a holomorphic function $f\colon B(0,r) \to \mathbb{C}$ such that $u = \Re(f)$. By Cauchy's integral formula, if γ is a parametrization of the positively oriented unit circle, then for all $w \in B(0,1)$ we have $f(w) = \frac{1}{2\pi i}\int_\gamma f(z)/(z-w)dz$, and so

$$u(z) = \Re\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w}\Big).$$

Since the integrand uses only the values of f on the boundary circle, we have almost recovered the function u from its values on the boundary. (Almost, because we appear to need the values of it harmonic conjugate). The next lemma resolves this:

Lemma 23.18. *If u is harmonic on $B(0,r)$ for $r > 1$ then for all $w \in B(0,1)$ we have*

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\Re\Big(\frac{e^{i\theta}+w}{e^{i\theta}-w}\Big)d\theta.$$

[56] The issue is whether the conformal equivalence behaves well enough at the boundaries.

Proof. (*Sketch.*) Take, as before, $f(z)$ holomorphic with $\Re(f) = u$ on $B(0,r)$. Then letting γ be a parametrization of the positively oriented unit circle we have

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-\bar{w}^{-1}}$$

where the first term is $f(w)$ by the integral formula and the second term is zero because $f(z)/(z-\bar{w}^{-1})$ is holomorphic inside all of $B(0,1)$. Gathering the terms, this becomes

$$f(w) = \frac{1}{2\pi}\int_\gamma f(z)\frac{1-|w|^2}{|z-w|^2}\frac{dz}{iz} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta.$$

The advantage of this last form is that the real and imaginary parts are now easy to extract, and we see that

$$u(z) = \int_0^{2\pi} u(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta.$$

Finally for the second integral expression note that if $|z| = 1$ then

$$\frac{z+w}{z-w} = \frac{(z+w)(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-|w|^2+(\bar{z}w-z\bar{w})}{|z-w|^2}.$$

from which one readily sees the real part agrees with the corresponding factor in our first expression. □

Now the idea to solve the Dirichlet problem for the disk $B(0,1)$ is to turn this previous result on its head: Notice that it tells us the values of u inside the disk $B(0,1)$ in terms of the values of u on the boundary. Thus if we are given the boundary values, say a (periodic) function $G(e^{i\theta})$ we might reasonably hope that the integral

$$g(w) = \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta,$$

would define a harmonic function with the required boundary values. Indeed it follows from the proof of the lemma that the integral is the real part of the integral

$$\frac{1}{2\pi i}\int_\gamma G(z)\frac{1}{z-w}dz,$$

which we know from Proposition 17.7 is holomorphic in w, thus $g(w)$ is certainly harmonic. It turns out that if $w \to w_0 \in \partial B(0,1)$ then provided G is continuous at w_0 then $g(w) \to G(w_0)$, hence g is in fact a harmonic function with the required boundary value.