In this appendix we review some notions from multivariable calculus. While we give careful proofs, only the statements are examinable.

26.1. **Properties of the Limit Superior.** We collect here some basic facts about the lim sup of a sequence of real numbers. Recall the definition:

Definition 26.1. Let (a_n) be a sequence which is bounded above (if it is not, by convention we set $\limsup_n(a_n) = +\infty$). Then for each n we may set $s_n = \sup\{a_k : k \geq n\}$. Clearly the sequence (s_n) is decreasing, and so if it is bounded below it has a limit, which we denote by $\limsup_n(a_n)$. If the sequence s_n is not bounded below, it tends to $-\infty$, and we write $\limsup_n(a_n) = -\infty$. Note that $\limsup_n(a_n) = -\infty$ if and only if $a_n \to -\infty$ as $n \to -\infty$.

The following Lemma is helpful in understanding what the properties of the lim sup are.

Lemma 26.2. *Let (a_n) be a sequence of real numbers which is bounded above and let $s = \limsup_n(a_n)$. If (a_{n_k}) is any convergent subsequence of (a_n) with limit ℓ then $\ell \leq s$. Moreover, there exists a subsequence of (a_n) which converges to s, so that $\limsup_n(a_n)$ is the maximum value of the limit of a subsequence of (a_n).*

Proof. For the first part, note that by definition clearly $a_{n_k} \leq s_{n_k}$, and since (s_n) tends to s it follows the subsequence (s_{n_k}) does also, hence since limits preserve weak inequalities, $\lim_k(a_{n_k}) = l \leq s$ as required.

Let $A_n = \{a_m : m \geq n \in \mathbb{N}\}$ be the set of values of the n-th tail of the sequence (a_n). Then it is clear that s_m is in $\bar{A}_n$ for each $m \geq n$, and so $s \in \bar{A}_n$ for all n. If s is a limit point of any A_n then it is easy to see that s is a limit of a subsequence of the associated tail $(a_k)_{k\geq n}$. If, for all n, we have $s \notin A'_n$, then we must have $s \in \bar{A}_n \backslash A'_n \subseteq A_n$ for all n, hence $s = a_m$ for infinitely many m. It follows that there is a subsequence of (a_n) which is constant and equal to s, so certainly it converges to s.

□

We have the following basic property of lim sup, which we used in the discussion of differentiation of power series:

Lemma 26.3. *Suppose that (a_n) is a bounded sequence of real numbers. Then if (c_n) is a sequence which converges to $c \geq 0$ then $\limsup_n(c_n a_n) = c.\limsup_n a_n$.*

Proof. If (a_{n_k}) is any subsequence of (a_n) which converges to $\ell \in \mathbb{R}$, then clearly $c_{n_k} a_{n_k} \to c.\ell$ as $n \to \infty$. Since $c \geq 0$ it follows the result follows from the previous lemma which shows that $\limsup_n(c_n a_n)$ is the maximum value of the limit of a subsequence of $(c_n a_n)$. □

Remark 26.4. For sequences which are bounded below one may consider $l_n = \inf\{a_k : k \geq n\}$. Clearly (l_n) forms an increasing sequence and one sets $\liminf_n(a_n) = \lim_n l_n$. It is easy to see that $\limsup_n(a_n) = -\liminf_n(-a_n)$.

26.2. **Partial derivatives and the total derivative.**

Theorem 26.5. *Suppose that $F: U \to \mathbb{R}^2$ is a function defined on an open subset of $\mathbb{R}^2$, whose partial derivatives exist and are continuous on U. Then for all $z \in U$ the function F is real-differentiable, with derivative Df_z given by the matrix of partial derivative.*

Proof. Working component by component, you can check that it is in fact enough to show that a function $f: U \to \mathbb{R}$ with continuous partial derivatives $\partial_x f$ and $\partial_y f$ has total derivative given by $(\partial_x f, \partial_y f)$ at each $z \in U$. That is, if $z = (x, y)$ then

$$f(x+h, y+k) = f(x, y) + \partial_x f(x, y)h + \partial_y f(x, y)k + \|(h, k)\|.\epsilon(h, k),$$

where $\epsilon(h, k) \to 0$ as $(h, k) \to 0$. But now since the function $x \mapsto f(x, y)$ is differentiable at x with derivative $\partial_x f(x, y)$ we have

$$f(x+h, y) = f(x, y) + \partial_x f(x, y)h + h\epsilon_1(h)$$

where $\epsilon_1(h) \to 0$ as $h \to 0$. Now by the mean value theorem applied the function to $y \mapsto f(x+h,y)$ we have

$$f(x+h,y+k) = f(x+h,y) + \partial_y f(x+h,y+\theta_2 k)k,$$

for some $\theta_2 \in (0,1)$. Thus using the definition of $\partial_x f(x,y)$ it follows that

$$f(x+h,y+k) = f(x,y) + \partial_x f(x,y)h + h\epsilon_1(h) + \partial_y f(x+h,y+\theta_2 k)k.$$

Thus we have

$$f(x+h,y+k) = f(x,y) + \partial_x f(x,y)h + \partial_y f(x,y)k + \|(h,k)\|\epsilon(h,k),$$

where

$$\epsilon(h,k) = \frac{h}{\sqrt{h^2+k^2}}\epsilon_1(h) + \frac{k}{\sqrt{h^2+k^2}}(\partial_y f(x+h,y+\theta_2 k) - \partial_y f(x,y)).$$

Thus since $0 \leq h/\sqrt{h^2+k^2}, k/\sqrt{h^2+k^2} \leq 1$, the fact that $\epsilon_1(h) \to 0$ as $h \to 0$ and the continuity of $\partial_y f$ at (x,y) imply that $\epsilon(h,k) \to 0$ as $(h,k) \to 0$ as required. □

Remark 26.6. Note that in fact the proof didn't use the full strength of the hypothesis of the theorem – we only actually needed the existence of the partial derivatives and the continuity of one of them at (x,y) to conclude that f is real-differentiable at (x,y).

26.3. **The Chain Rule.** We establish a version of the chain rule which is needed for the proof that the existence of a primitive for a function $f\colon U \to \mathbb{C}$ implies that $\int_\gamma f(z)dz = 0$ for every closed curve γ in U. The proof requires one to use the fact that if $dF/dt = f$ on U then $f(\gamma(t))\gamma'(t)$ is the derivative of $F(\gamma(t))$. This is of course formally exactly what one would expect using the formula for the normal version of the chain rule, but one should be slightly careful: $F\colon \mathbb{C} \to \mathbb{C}$ is a function of a complex variable, while $\gamma\colon [a,b] \to \mathbb{C}$ is a function of real variable, so we are mixing real and complex differentiability.

That said, we have seen that a complex differentiable function is also differentiable in the real sense, with its derivative being the linear map given by multiplication by the complex number which is its complex derivative. Thus the result we need follows from a version of the chain rule for real-differentiable functions:

Lemma 26.7. *Let U be an open subset of $\mathbb{R}^2$ and let $F\colon U \to \mathbb{R}^2$ be a differentiable function. If $\gamma\colon [a,b] \to \mathbb{R}$ is a (piecewise) C^1-path with image in U, then $F(\gamma(t))$ is a differentiable function with*

$$\frac{d}{dt}(F(\gamma(t))) = DF_{\gamma(t)}(\gamma'(t))$$

Proof. Let $t_0 \in [a,b]$ and let $z_0 = \gamma(t_0) \in U$. Then by definition, there is a function $\epsilon(z)$ such that

$$F(z) = F(z_0) + DF_{z_0}(z-z_0) + |z-z_0|\epsilon(z),$$

where $\epsilon(z) \to 0 = \epsilon(z_0)$ as $z \to z_0$. But then

$$\frac{F(\gamma(t)) - F(\gamma(t_0))}{t-t_0} = DF_{z_0}(\frac{\gamma(t)-\gamma(t_0)}{t-t_0}) + \epsilon(\gamma(t)).\frac{|\gamma(t)-\gamma(t_0)|}{t-t_0}.$$

But now consider the two terms on the right-hand side of this expression: for the first term, note that a linear map is continuous, so since $(\gamma(t)-\gamma(t_0))/(t-t_0) \to \gamma'(t_0)$ as $t \to t_0$ we see that $DF_{z_0}(\frac{\gamma(t)-\gamma(t_0)}{t-t_0}) \to DF_{z_0}(\gamma'(t_0))$ as $t \to t_0$. On the other hand, for the second term, since $\frac{\gamma(t)-\gamma(t_0)}{t-t_0}$ tends to $\gamma'(t_0)$ as t tends to t_0, we see that $|\gamma(t)-\gamma(t_0)|/(t-t_0)$ is bounded as $t \to t_0$, while since $\gamma(t)$ is continuous at t_0 since it is differentiable there $\epsilon(\gamma(t)) \to \epsilon(\gamma(t_0)) = \epsilon(z_0) = 0$. It follows that the second term tends to zero, so that the left-hand side tends to $Df_{\gamma(t_0)}(\gamma'(t_0))$ as required. □

Remark 26.8. Notice that the proof above works in precisely the same way if F is a function from $\mathbb{R}^2$ to $\mathbb{R}$. Indeed a slight modification of the argument proves that if $F\colon \mathbb{R}^n \to \mathbb{R}^m$ and $G\colon \mathbb{R}^m \to \mathbb{R}^p$ then if F and G are differentiable, their composite $G \circ F$ is differentiable with derivative $DG_{F(x)} \circ DF_x$.

An easy application of the chain rule is the following constancy theorem. For the proof it is convenient to introduce some terminology:

Definition 26.9. We say a function $f\colon X \to Y$ between metric spaces is *locally constant* if for any $z \in X$ there is an $r > 0$ such that f is constant on $B(z, r)$.

Remark 26.10. Clearly a locally constant function is continuous, and moreover for such a function, the pre-image of any point in its image is an open set. Since for any continuous function the pre-image of a point is a closed set, it follows the pre-image of a point in the range of a locally-constant function is both open and closed. Thus if X is connected and f is locally constant, then f is in fact constant.

Proposition 26.11. *Suppose that $f\colon U \to \mathbb{R}^2$ is a function defined on a connected open subset of $\mathbb{R}^2$. Then if $Df_z = 0$ for all $z \in U$ the function f is constant.*

Proof. By the preceding remarks it suffices to show that f is locally constant. To see this, let $z_0 \in U$ and fix $r > 0$ such that $B(z_0, r) \subseteq U$. Then for any $z \in B(z_0, r)$ we may consider the function $F(t) = f(z_0 + t(z - z_0))$, where $t \in [0, 1]$. Note that $F = f \circ \gamma$ where $\gamma(t) = z_0 + t(z - z_0)$ is the straight line-segment from z_0 to z which lies entirely in $B(z_0, r)$ as z does. Hence applying the chain rule we have $F'(t) = Df_{z_0 + t(z - z_0)}(z - z_0) = 0$ by our assumption on Df_z. It follows from the Fundamental Theorem of Calculus that

$$f(z) - f(z_0) = F(1) - F(0) = \int_0^1 F'(t)\,dt = 0,$$

hence f is constant on $B(z_0, r)$ as required. (The integral of $F'(t) = (u'(t), v'(t))$ is taken component-wise □

26.4. **Symmetry of mixed partial derivatives.** We used in the proof that the real and imaginary parts of a holomorphic function are harmonic the fact that partial derivatives commute on twice continuously differentiable functions. We give a proof of this for completeness. The key to the proof will be to use difference operators:

Definition 26.12. Let $f\colon U \to \mathbb{R}$ be a function defined on an open set $U \subset \mathbb{R}^2$. Then if $s, t \in \mathbb{R}\backslash\{0\}$ let $\Delta_1^s(f), \Delta_2^t(f)$ be the function given by

$$\Delta_1^s(f)(x, y) = \frac{f(x + s, y) - f(x, y)}{s}, \quad \Delta_2^t(f)(x, y) = \frac{f(x, y + t) - f(x, y)}{t}$$

Note that if f is differentiable at (x, y) then $\partial_x f(x, y) = \lim_{s \to 0} \Delta_1^s(f)(x, y)$ and $\partial_y f(x, y) = \lim_{t \to 0} \Delta_2^t(f)(x, y)$.

It is straight-forward to check that

$$\begin{aligned}\Delta_1^2(\Delta_2^t(f))(x, y) &= \Delta_2^t(\Delta_1^s(f))(x, y)\\ &= \frac{f(x + s, y + t) - f(x + s, y) - f(x, y + t) + f(x, y)}{st}.\end{aligned}$$

That is, the two difference operators $f \mapsto \Delta_1^s(f)$ and $f \mapsto \Delta_2^t(f)$ commute with each other. We wish to use this fact to deduce that the corresponding partial differential operators also commute, but because of the limits involved, this will not be automatic, and we will need to impose the additional hypotheses that the second partial derivatives of f are continuous functions.

Since the difference operator Δ_1^s and Δ_2^t are linear, they commute with partial differentiation so that $\partial_y \Delta_1^s(f)(x, y) = \Delta_1^s(\partial_y f)(x, y)$, and similarly for ∂_x and also for Δ_2^t and ∂_x, ∂_y.

We are now ready to prove that mixed partial derivatives are equal:

Lemma 26.13. *Suppose that $f\colon U \to \mathbb{R}$ is twice continuously differentiable, so that all its second partial derivatives exist and are continuous on U. Then*

$$\partial_x \partial_y f = \partial_y \partial_x f$$

on U.

Proof. Fix $(x, y) \in U$. Since U is open, there are $\epsilon, \delta > 0$ such that $\Delta_1^s(f)$ and $\Delta_2^t(f)$ are defined on $B((x, y), \epsilon)$ for all s, t with $|s|, |t| < \delta$. Now by definition we have

$$\partial_x \partial_y f(x, y) = \partial_x (\lim_{t \to 0} \Delta_2^t(f))(x, y) = \lim_{s \to 0} \lim_{t \to 0} \Delta_1^s \Delta_2^t(f)(x, y)$$

But now using the mean value theorem for $\Delta_2^t(f)$ in the first variable, we see that

$$\Delta_1^s \Delta_2^t(f)(x, y) = \partial_x \Delta_2^t f(x + s_1, y),$$

where s_1 lies between 0 and s. But $\partial_x \Delta_2^t(f)(x+s_1, y) = \Delta_2^t \partial_x f(x+s_1, y)$, and using the mean value theorem for $\partial_x f(x + s_1, y)$ in the second variable we see that $\Delta_2^t \partial_x f(x + s_1, y) = \partial_y \partial_x f(x + s_1, y + t_1)$ where t_1 lies between 0 and t (and note that t_1 depends both on t and s_1).

But now

$$\partial_x \partial_y f(x, y) = \lim_{s \to 0} \lim_{t \to 0} \partial_y \partial_x f(x + s_1, y + t_1) = \partial_y \partial_x f(x, y),$$

by the continuity of the second partial derivatives, so we are done.

□

Example 26.14. Let $\Delta = \partial_x^2 + \partial_y^2$ be the (two-dimensional) Laplacian. Provided we are only interested in acting on twice-continuously differentiable functions $u = u(x, y)$ so that $\partial_x \partial_y(u) = \partial_y \partial_x(u)$, we can factorize Δ as

$$\Delta = (\partial_x - i\partial_y)(\partial_x + i\partial_y).$$

This is the key to the relationship between holomorphic and harmonic functions.