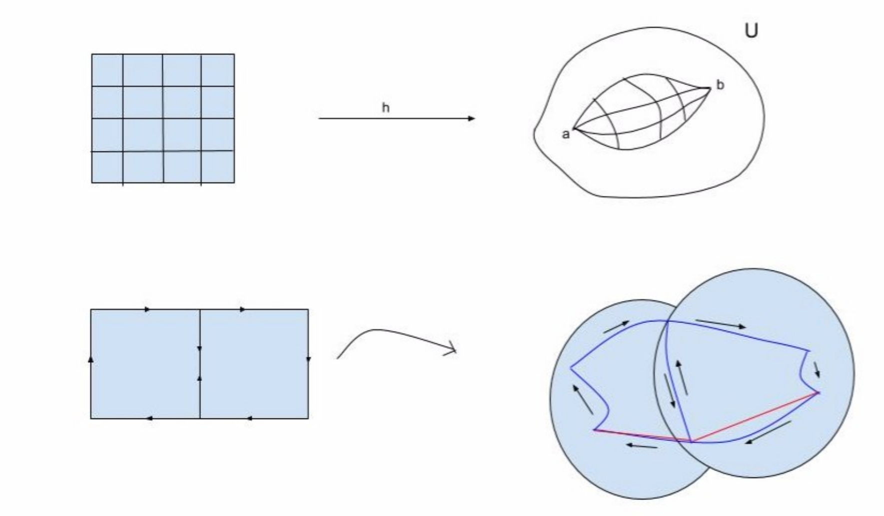

FIGURE 6. Dissecting the homotopy

28. APPENDIX IV: ON THE HOMOTOPY AND HOMOLOGY VERSIONS OF CAUCHY'S THEOREM

In this appendix we give proofs of the homotopy and homology versions of Cauchy's theorem which are stated in the body of the notes. These proofs are non-examinable, but are included for the sake of completeness.

Theorem 28.1. *Let U be a domain in $\mathbb{C}$ and $a, b \in U$. Suppose that γ and η are paths from a to b which are homotopic in U and $f\colon U \to \mathbb{C}$ is a holomorphic function. Then*

$$\int_\gamma f(z)dz = \int_\eta f(z)dz.$$

Proof. The key to the proof of this theorem is to show that the integrals of f along two paths from a to b which "stay close to each other" are equal. We show this by covering both paths by finitely many open disks and using the existence of a primitive for f in each of the disks.

More precisely, suppose that $h\colon [0,1]\times[0,1]$ is a homotopy between γ and η. Let us write $K = h([0,1]\times[0,1])$ be the image of the map h, a compact subset of U. By Lemma 11.6 there is an $\epsilon > 0$ such that $B(z,\epsilon) \subseteq U$ for all $z \in K$.

Next we use the fact that, since $[0,1]\times[0,1]$ is compact, h is uniformly continuous. Thus we may find a $\delta > 0$ such that $|h(t_1,s_1) - h(t_2,w_2)| < \epsilon$ whenever $\|(t_1,s_1)-(t_2,s_2)\| < \delta$. Now pick $N \in \mathbb{N}$ such that $1/N < \delta$ and dissect the square $[0,1]\times[0,1]$ into N^2 small squares of side length $1/N$. For convenience, we will write $t_i = i/N$ for $i \in \{0,1,\dots,N\}$

For each $k \in \{1,2,\dots,N-1\}$, let ν_k be the piecewise linear path which connects the point $h(t_j,k/N)$ to $h(t_{j+1},k/N)$ for each $j \in \{0,1,\dots,N)$. Explicitly, for $t \in [t_j,t_{j+1}]$, we set

$$\nu_k(t) = h(t_j,k/N)(1-Nt-j) + h(t_{j+1},k/N)(Nt-j)$$

We claim that

$$\int_\gamma f(z)dz = \int_{\nu_1} f(z)dz = \int_{\nu_2} f(z)dz = \ldots = \int_{\nu_{N-1}} f(z)dz = \int_\eta f(z)dz$$

which will prove the theorem. In fact, we will only show that $\int_\gamma f(z)dz = \int_{\nu_1} f(z)dz$, since the other cases are almost identical.

We may assume the numbering of our squares S_i is such that $S_1,\dots,S_N$ list the bottom row of our N^2 squares from left to right. Let m_i be the centre of the square S_i and let $p_i = h(m_i)$. Then $h(S_i) \subseteq B(p_i,\epsilon)$ so that $\gamma([t_i, t_{i+1}]) \subseteq B(p_i,\epsilon)$ and $\nu_1([t_i, t_{i+1}]) \subseteq B(p_i,\epsilon)$ (since $B(p_i,\epsilon)$ is convex and by assumption contains $\nu_1(t_i)$ and $\nu_1(t_{i+1})$). Since $B(p_i,\epsilon)$ is convex, f has primitive F_i on each $B(p_i,\epsilon)$. Moreover, as primitives of f on a domain are unique up to a constant, it follows that F_i and F_{i+1} differ by a constant on $B(p_i,\epsilon) \cap B(p_{i+1},\epsilon)$, where they are both defined. In particular, since $\gamma(t_i), \nu_1(t_i) \in B(p_i,\epsilon) \cap B(p_{i+1},\epsilon)$, $(1 \le i \le N-1)$, we have

$$F_i(\gamma(t_i)) - F_{i+1}(\gamma(t_i)) = F_i(\nu_1(t_i)) - F_{i+1}(\nu_1(t_i)). \tag{28.1}$$

Now by the Fundamental Theorem we have

$$\int_{\gamma_{|[t_i,t_{i+1}]}} f(z)dz = F_i(\gamma(t_{i+1})) - F_i(\gamma_1(t_i)),$$
$$\int_{\nu_{1|[t_i,t_{i+1}]}} f(z)dz = F_i(\nu_1(t_{i+1})) - F_i(\nu_1(t_i))$$

Combining we find that:

$$\begin{aligned}
\int_\gamma f(z)dz &= \sum_{i=0}^{N-1} \int_{\gamma_{|[t_i,t_{i+1}]}} f(z)dz \\
&= \sum_{i=0}^{N-1} \big(F_{i+1}(\gamma(t_{i+1})) - F_{i+1}(\gamma(t_i))\big) \\
&= F_N(\gamma(t_N)) - F_1(\gamma(0)) + \sum_{i=1}^{N-1} \big(F_i(\gamma(t_i)) - F_{i+1}(\gamma(t_i))\big) \\
&= F_N(b) - F_0(a) + \Big(\sum_{i=0}^{N-1} (F_i(\nu_1(t_{i+1})) - F_{i+1}(\nu_1(t_{i+1})\Big) \\
&= \sum_{i=0}^{N-1} \big((F_{i+1}(\nu_1(t_{i+1})) - F_{i+1}(\nu_1(t_i))\big) \\
&= \sum_{i=0}^{N-1} \int_{\nu_{1|[t_i,t_{i+1}]}} f(z)dz = \int_{\nu_1} f(z)dz
\end{aligned}$$

where in the fourth equality we used Equation (28.1).

□

Remark 28.2. The use of the piecewise linear paths ν_k might seem unnatural – it might seem simpler to use the paths given by the homotopy, that is the paths $\gamma_k(t) = h(t, k/N)$. The reason we did not do this is because we only assume that h is continuous, so we do not know that the path γ_k is piecewise C^1 which we need in order to be able to integrate along it.

The proof of the homology form of Cauchy's theorem uses Liouville's theorem, which we proved using Cauchy's theorem for a disc.

Theorem 28.3. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma\colon [0,1] \to U$ be a closed path whose inside lies entirely in U, that is $I(\gamma, z) = 0$ for all $z \notin U$. Then we have, for all $z \in U\backslash\gamma^*$,*

$$\int_\gamma f(\zeta)d\zeta = 0; \quad \int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = 2\pi i I(\gamma, z) f(z), \quad \forall z \in U\backslash\gamma^*.$$

Moreover, if U is simply-connected and $\gamma\colon [a,b] \to U$ is any closed path, then $I(\gamma, z) = 0$ for any $z \notin U$, so the above identities hold for all closed paths in such U.

Proof. We first prove the general form of the integral formula. Note that using the integral formula for the winding number and rearranging, we wish to show that

$$F(z) = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = 0$$

for all $z \in U \backslash \gamma^*$. Now if $g(\zeta, z) = (f(\zeta) - f(z))/(\zeta - z)$, then since f is complex differentiable, g extends to a continuous function on $U \times U$ if we set $g(z, z) = f'(z)$. Thus the function F is in fact defined for all $z \in U$. Moreover, if we fix ζ then, by standard properties of differentiable functions, $g(\zeta, z)$ is clearly complex differentiable as a function of z everywhere except at $z = \zeta$. But since it extends to a continuous function at ζ, it is bounded near ζ, hence by Riemann's removable singularity theorem, $z \mapsto g(\zeta, z)$ is in fact holomorphic on all of U. It follows by Theorem 18.22 that

$$F(z) = \int_0^1 g(\gamma(t), z)\gamma'(t) dt$$

is a holomorphic function of z.

Now let $\text{ins}(\gamma) = \{z \in \mathbb{C} : I(\gamma, z) \neq 0\}$ be the inside of γ, so by assumption we have $\text{ins}(\gamma) \subset U$, and let $V = \mathbb{C} \backslash (\gamma^* \cup \text{ins}(\gamma))$ be the complement of γ^* and its inside. If $z \in U \cap V$, that is, $z \in U$ but not inside γ or on γ^*, then

$$\begin{aligned} F(z) &= \int_\gamma \frac{f(\zeta) d\zeta}{\zeta - z} - f(z) \int_\gamma \frac{d\zeta}{\zeta - z} \\ &= \int_\gamma \frac{f(\zeta) d\zeta}{\zeta - z} - f(z) I(\gamma, z) \\ &= \int_\gamma \frac{f(\zeta) d\zeta}{\zeta - z} = G(z) \end{aligned}$$

since $I(\gamma, z) = 0$. Now $G(z)$ is an integral which only involves the values of f on γ^* hence it is defined for all $z \notin \gamma^*$, and by Theorem 18.22, $G(z)$ is holomorphic. In particular G defines a holomorphic function on V, which agrees with F on all of $U \cap V$, and thus gives an extension of F to a holomorphic function on all of $\mathbb{C}$. (Note that by the above, F and G will in general *not* agree on the inside of γ.) Indeed if we set $H(z) = F(z)$ for all $z \in U$ and $H(z) = G(z)$ for all $z \in V$ then H is a well-defined holomorphic function on all of $\mathbb{C}$. We claim that $|H| \to 0$ as $|z| \to \infty$, so that by Liouville's theorem, $H(z) = 0$, and so $F(z) = 0$ as required. But since $\text{ins}(\gamma)$ is bounded, there is an $R > 0$ such that $V \supseteq \mathbb{C} \backslash B(0, R)$, and so $H(z) = G(z)$ for $|z| > R$. But then setting $M = \sup_{\zeta \in \gamma^*} |f(\zeta)|$ we see

$$|H(z)| = \left| \int_\gamma \frac{f(\zeta) d\zeta}{\zeta - z} \right| \leq \frac{\ell(\gamma).M}{|z| - R}.$$

which clearly tends to zero as $|z| \to \infty$, hence $|H(z)| \to 0$ as $|z| \to \infty$ as required.

For the second formula, simply apply the integral formula to $g(z) = (z - w) f(z)$ for any $w \notin \gamma^*$. Finally, to see that if U is simply-connected the inside of γ always lies in U, note that if $w \notin U$ then $1/(z - w)$ is holomorphic on all of U, and so $I(\gamma, w) = \int_\gamma \frac{dz}{z-w} = 0$ by the homotopy form of Cauchy's theorem. □

Remark 28.4. It is often easier to check a domain is simply-connected than it is to compute the interior of a path. Note that the above proof uses Liouville's theorem, whose proof depends on Cauchy's Integral Formula for a circular path, which was a consequence of Cauchy's theorem for a triangle, but apart from the final part of the proof on simply-connectd regions, we did not use the more sophisticated homotopy form of Cauchy's theorem. We have thus established the winding number and homotopy forms of Cauchy's theorem essentially independently of each other.