29. Appendix V: Remark on the Inverse Function Theorem

In this appendix we supply[59] *the details for the claim made in the remark after the proof of the holomorphic version of the inverse function theorem.*

There is an enhancement of the Inverse Function Theorem in the holomorphic setting, which shows that the condition $f'(z) \neq 0$ is automatic (in contrast to the case of real differentiable functions, where it is essential as one sees by considering the example of the function $f(x) = x^3$ on the real line). Indeed suppose that $f: U \to \mathbb{C}$ is a holomorphic function on an open subset $U \subset \mathbb{C}$, and that we have $z_0 \in \mathbb{U}$ such that $f'(z_0) = 0$.

Claim: In this case, f is at least 2 to 1 near z_0, and hence is not injective.

Proof of Claim: If we let $w_0 = f(z_0)$ and $g(z) = f(z) - w_0$, it follows g has a zero at z_0, and thus it is either identically zero on the connected component of U containing z_0 (in which case it is very far from being injective!) or we may write $g(z) = (z - z_0)^k h(z)$ where $h(z)$ is holomorphic on U and $h(z_0) \neq 0$. Our assumption that $f'(z_0) = 0$ implies that k, the multiplicity of the zero of g at z_0 is at least 2.

Now since $h(z_0) \neq 0$, we have $\epsilon = |h(z_0)| > 0$ and hence by the continuity of h at z_0 we may find a $\delta > 0$ such that $h(B(z_0,\delta)) \subseteq B(h(z_0),\epsilon)$. But then by taking a cut along the ray $\{-t.h(z_0) : t \in \mathbb{R}_{>0}\}$ we can define a holomorphic branch of $z \mapsto z^{1/k}$ on the whole of $B(h(z_0),\epsilon)$. Now let $\phi\colon B(z_0,\delta) \to \mathbb{C}$ be the holomorphic function given by $\phi(z) = (z - z_0).h(z)^{1/k}$ (where by our choice of δ this is well-defined) so that $\phi'(z_0) = h(z_0)^{1/k} \neq 0$. Then clearly $f(z) = w_0 + \phi(z)^k$ on $B(z_0,\delta)$. Since $\phi(z)$ is holomorphic,the open mapping theorem ensures that $\phi(B(z_0,\delta))$ is an open set, which since it contains $0 = \phi(z_0)$, contains $B(0,r)$ for some $r > 0$. But then since $z \mapsto z^k$ is k-to-1 as a map from $B(0,r)\backslash\{0\} \to B(0,r^k)\backslash\{0\}$ it follows that f takes every value in $B(w_0,r^k)\backslash\{w_0\}$ at least k times.

[59] For interest, not examination!