A1 Differential Equations I: MT 2019/20: Sheet 1.

Material on uniformly convergent sequences and series, needed in this course and covered in Prelims is summarised in questions 1 and 2. These should be revision and should be done before the lectures. Questions 3, 4, and 5 are based on material in lectures.

1.1 Let $[a, b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n\geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a, b]$. What does it mean to say that **the sequence converges uniformly on** $[a, b]$ **to a limit function** y**?** If each y_n is continuous on $[a, b]$ show that the uniform limit y is continuous on $[a, b]$ and that, when $n \to \infty$,

$$\int_a^b |y_n(x) - y(x)|dx \to 0, \qquad \int_a^b y_n(x)dx \to \int_a^b y(x)dx.$$

If $[a, b] = [0, 1]$ and $y_n(x) = nxe^{-nx^2}$ show that, for each $x \in [0, 1], y_n(x) \to 0$ but $\displaystyle\int_0^1 y_n(x)dx \to \frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0\leq x\leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.

1.2 Let $\displaystyle\sum_{n=0}^{\infty} u_n$ be a series of real-valued functions defined on $[a, b]$. State the **Weierstrass M-test** for the uniform convergence of the series.

Show that the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{1+n^2}$ converges uniformly on $[-\pi, \pi]$.

ODEs and Picard's Theorem:

1.3 Consider the initial-value problems

$$\begin{aligned} y' &= x^2 + y^2, \quad y(0) = 0, \qquad (1)\\ y' &= (1 - 2x)y, \quad y(0) = 1. \qquad (2) \end{aligned}$$

In each case find y_0, y_1, y_2, y_3, where $\{y_n\}_{n\geq 0}$ is the sequence of Picard approximations. By considering the behaviour of x^2+y^2 on the square $\{(x, y) : |x| \leq \frac{1}{\sqrt{2}},\ |y| \leq \frac{1}{\sqrt{2}}\}$ and appealing to Picard's theorem show that in case (1) the sequence converges uniformly for $|x| \leq \frac{1}{\sqrt{2}}$.

In case (2), use Picard's theorem to show that the problem has a unique solution for all x. Now find the solution explicitly and, by expanding as a series, show that the sequence $\{y_n\}_{n\geq 0}$ converges to the solution.

P.T.O.

1.4 Consider the initial-value problem

$$y' = xy^{1/3}, \quad y(0) = b,$$

a) (i) Does the function $F(x, y) = xy^{1/3}$ satisfy a Lipschitz condition on the rectangle $\{(x, y) : |x| \leq h,\ |y| \leq k\}$, where $h > 0$ and $k > 0$?

(ii) If $b > 0$ use Picard's theorem to show that there is a unique solution on an interval $[-h, h]$, for a suitable $h > 0$ which you should specify (you must check carefully that the assumptions of Picard's theorem are satisfied).

iii) If $b = 0$, show that for any $c > 0$ there is a solution y which is identically zero on $[-c, c]$ and positive when $|x| > c$.

b) [Optional] Now return to the case $b > 0$. Consider the set $R = \{(x, y) : y \geq b, |x| \leq h\}$. By working in this R, and adapting the proof of Picard's theorem, prove that in fact there is a unique solution of the problem on $|x| \leq h$ for any h and hence that there is global existence of solutions.

1.5 Suppose that $f : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \to \mathbb{R}$ are continuous. Consider the integral equation for $y(x)$

$$y(x) = f(x) + \int_a^x K(x, t)y(t)dt, \quad x \in [a, b].$$

For $x \in [a, b]$ define

$$\begin{aligned} y_0(x) &= f(x) \\ y_{n+1}(x) &= f(x) + \int_a^x K(x, t)y_n(t)dt. \end{aligned}$$

Adapt the proof of Picard's theorem to show that y_n converges uniformly to a solution of the integral equation for all $x \in [a, b]$. [You may assume that if $y : [a, b] \to \mathbb{R}$ is continuous then so too is $f(x) + \int_a^x K(x, t)y(t)dt$ for $x \in [a, b]$.]

Now show that the solution is unique.

Prove also that the solution depends continuously on f. [You will need to decide what this means.]