A1 Differential Equations I: MT 2019/20 Sheet 2.

Systems of non-linear ODEs.

2.1 The aim of this question is to fill in the details of the proof of Theorem 1.6 in the lecture notes of Picard's theorem for a system of two first order ODEs via the CMT.

Consider the system of first order ODEs, for the functions y_1 and y_2

$$y_1'(x) = f_1(x, y_1(x), y_2(x)) \tag{1}$$

$$y_2'(x) = f_2(x, y_1(x), y_2(x)) \tag{2}$$

$$\text{with initial condition} \quad y_1(a) = b_1, \quad y_2(a) = b_2. \tag{3}$$

If we write

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$

then we can write the problem (1), (2), (3) in vector form as

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x)), \tag{4}$$

$$\underline{y}(a) = \underline{b}, \tag{5}$$

We will use the l^1 norm in $\mathbb{R}^2$, $||(y_1, y_2)||_1 = |y_1| + |y_2|$. Let $B_k(\underline{b})$ be the disc in $\mathbb{R}^2$, centre $\underline{b}$, radius k. Define the set $S = \{(x, \underline{y}) \in \mathbb{R}^3 : |x-a| \le h, \ \underline{y} \in B_k(\underline{b})\}$. We assume $\underline{f}$ is continuous on the set S, with $\sup_S ||\underline{f}(x, \underline{y})||_1 \le M$, and for $x \in [a-h, a+h]$, $\underline{f}(x, \underline{y})$ is Lipschitz continuous with respect to $\underline{y}$ on S. That is, there exists L such that for $x \in [a-h, a+h]$ and $\underline{u}, \underline{v} \in B_k(\underline{b})$,

$$||\underline{f}(x, \underline{u}) - \underline{f}(x, \underline{v})||_1 \le L||\underline{u} - \underline{v}||_1. \tag{6}$$

We will work in the space $\mathcal{C}_h = \mathcal{C}([a-h, a+h]; B_k(\underline{b}))$ of continuous functions from $[a-h, a+h]$ to the disc $B_k(\underline{b})$ in $\mathbb{R}^2$, with the sup norm defined for $\underline{y} \in C_h$ by

$$||\underline{y}||_{\sup} := \sup_{x \in [a-h, a+h]} ||\underline{y}(x)||_1.$$

We can write the initial value problem (4), (5) as an integral equation

$$\underline{y}(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds \tag{7}$$

where by the integral we mean that we integrate componentwise.

Now we define

$$(T\underline{y})(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds$$

so we can write equation (6) as a fixed point problem in $\mathcal{C}_\eta$, for $0 < \eta \le h$.

$$\underline{y} = T\underline{y}.$$

(i) Prove that for $\underline{g} \in \mathcal{C}_h$,

$$\left\| \int_a^x \underline{g}(t)dt \right\|_1 \leq \left| \int_a^x \|\underline{g}(t)\|_1 \, dt \right|.$$

[You may assume that if $h : [a, x] \to \mathbb{R}$ is continuous then $\left|\int_a^x h(t)dt\right| \leq \left|\int_a^x |h(t)|\, dt\right|$.]

(ii) Prove that for suitable $0 < \eta \leq h$, T satisfies the conditions of the CMT so has a unique fixed point. Explain why this solution is also the unique solution of (4), (5).

[You may assume that $\mathcal{C}_h$ is a complete metric space]

(iii) Explain why, if the Lipschitz condition (6) holds for all $x \in [a-h, a+h]$ and $\underline{u}, \underline{v} \in \mathbb{R}^2$ (instead of only $\underline{u}, \underline{v} \in B_k(\underline{b})$), then there is a unique solution for all $x \in [a-h, a+h]$.

(iv) Show that solutions depend continuously on the initial data. (You will have to decide what this means precisely.)

2.2 (a) Consider the second order differential equation for $y(x)$

$$y''(x) = F(x, y(x), y'(x)), \quad x \in [a-h, a+h], \tag{8}$$

with initial conditions $y(a) = c$, $y'(a) = d$. Set $S = \{(x, u, v) : |x - a| \leq h, |u - c| + |v - d| \leq k\}$. Suppose that $F(x, u, v)$ is continuous on S and that there exists L such that at all points in S

$$|F(x, u_1, v_1) - F(x, u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|).$$

By writing (8) as a system of differential equations, and demonstrating that the conditions of Theorem 1.6 (Picard's existence theorem for systems, see question 2.1) are satisfied, show that there exists $0 < \eta \leq h$ such that (8) with the given initial conditions has a unique solution on $[a-\eta, a+\eta]$.

(b) Now consider the second order linear differential equation for $y(x)$

$$p(x)y'' + q(x)y' + r(x)y = s(x), \quad x \in [a, b] \tag{9}$$

with initial condition $y(x_0) = y_0$, $y'(x_0) = y_1$, with $x_0 \in [a, b]$. Here $p(x) \neq 0$ and p, q, r and s are continuous. Show that (9) with the given initial conditions has a unique solution on $[a, b]$.

(c)(Taken from Collins) Consider the problem

$$yy'' = -(y')^2, \quad y(0) = y'(0) = 1.$$

(i) Use part (a) to show that the problem has a unique solution on an interval containing 0.

(ii) Find the solution and state where it exists.

Autonomous systems of ODEs and the phase plane

2.3 Consider the plane autonomous system

$$\frac{dx}{dt} = x(1 - 2x - y), \quad \frac{dy}{dt} = y(1 - x - 2y).$$

By showing that the axes of the phase plane and the line $x = y$ are solution trajectories explain why a solution starting in the octant $x > 0$, $x < y$ must remain in this region for all time.

Find all the critical points and analyse them to determine their local behaviour including the local direction of the trajectories and whether the points are stable.

Sketch the phase plane.

Use the Bendixson-Dulac theorem, with $\phi = 1/xy$, to show that there are no closed trajectories in $R = \{(x, y) : x > 0,\ y > 0\}$.

In an application, when suitably scaled, x and y represent species populations which are in competition for resources. Use the phase plane to interpret what happens to the populations in the long term.

2.4 Find and classify the types of all critical points of the system

$$\frac{dx}{dt} = (a - x^2)y, \quad \frac{dy}{dt} = x - y,$$

in each of the cases (i) $a < -\frac{1}{4}$, (ii) $-\frac{1}{4} < a < 0$, (iii) $a > 0$.

Consider the case $a = -1/4$ and analyse in detail the behaviour at all the critical points. Hence sketch the phase plane in this case.

[Optional] Consider the case $a = 1/2$ and analyse in detail the behaviour at all the critical points. Hence sketch the phase plane in this case.