Part A Numerical Analysis, Hilary 2020. Problem Sheet 4

1. Verify that $\max_{x\in[a,b]} |f(x)|$ defines a norm on $C[a,b]$.

2. For each of the following, say if it defines a norm on $C^1[a,b]$ (the vector space of continuously differentiable functions on $[a,b]$), and if not, why not:

(i) $\left|\int_a^b f(x)\,\mathrm{d}x\right|$

(ii) $\max_{x\in[a,b]} |f(x)+f'(x)|$

(iii) $\max_{x\in[a,b]} [f(x)]^2$

(iv) $\max_{x\in[a,b]} \{|f(x)|+|f'(x)|\}$

3. In the inner product $\langle f,g\rangle = \int_0^2 xf(x)g(x)\,\mathrm{d}x$, calculate the angle between $1/\sqrt{x}$ and $3x$. Can you calculate the angle if the inner product is $\langle f,g\rangle = \int_0^2 f(x)g(x)\,\mathrm{d}x$?

4. Calculate the orthogonal polynomials ϕ_0, ϕ_1, ϕ_2 in the inner product space defined by

$$\langle f,g\rangle = \int_0^2 xf(x)g(x)\,\mathrm{d}x.$$

5. Calculate the best approximation to x^3 on $[0,2]$ from Π_2 in the norm derived from the inner product as above,

$$\int_0^2 xf(x)g(x)\,\mathrm{d}x = \langle f,g\rangle.$$

[If you like you can use MATLAB, which is very good at solving systems of linear equations as you know by now!]

6. By considering $\|f-(p+\epsilon q)\|^2$ where $\epsilon\in\mathbb{R}$, $q\in\Pi_n$, show that if $p\in\Pi_n$ is a best approximation to f in this norm with associated inner product $\langle\cdot,\cdot\rangle$ then $\langle f-p,q\rangle = 0$ for any $q\in\Pi_n$.

7. If $\{\phi_0,\phi_1,\dots,\phi_n,\dots\}$ are orthogonal polynomials in $\langle\cdot,\cdot\rangle$ which are normalised to be monic (i.e. have leading coefficient equal to 1) show that $\|\phi_k\|\le\|q\|$ for all monic polynomials $q\in\Pi_k$ which are of exact degree k where $\|\cdot\|$ is the norm derived from the inner product.

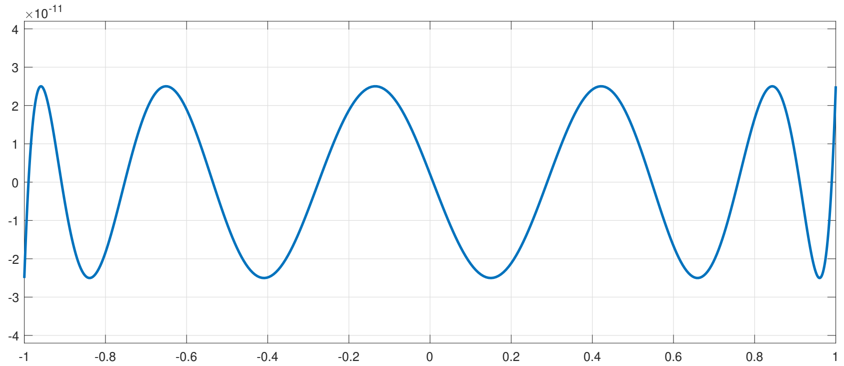
8. Let $\mu_j = \int_a^b x^j w(x)\,\mathrm{d}x$ be the jth *moment* of the weight distribution $w(x)$. Show that the linear system of equations

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{bmatrix}$$

has as solution the coefficients of a polynomial $x^n - \sum_{j=0}^{n-1} c_j x^j$, which is a member of the family of orthogonal polynomials associated with the weight function w.

9. Let $p(x) = \sum_{k=0}^{n} c_k \phi_k(x)$ where $\{\phi_k\}_{k=0}^{n}$ are the *orthonormal* Legendre polynomials on $[-1, 1]$. (i) What is $\int_{-1}^{1} p(x)dx$? (ii) What is the best degree-k polynomial approximant to p in the L_2-norm? (i.e., minimiser of $\int_{-1}^{1}(p(x) - q_k(x))^2 dx$ over $q_k \in \Pi_k$)

10. Let $f : [a, b] \to \mathbb{R}$ be a real continuous function. Consider finding the best degree-k polynomial approximant p_k to f on $[a, b]$ in the L_∞-norm (also known as minimax approximation). The solution is known to have a beautiful "equioscillation" property. For example, below is the error $\exp(x) - p_{10}(x)$ of the degree 10 minimax polynomial approximant to the exponential function on $[-1, 1]$.



Make this precise by proving that equioscillation implies optimality: If $f - p_k$ has $k+2$ extrema $(a \le) x_1 < x_2 < \cdots < x_{k+2} (\le b)$ with alternating signs, i.e., $f(x_i) - p_k(x_i) = (-1)^{i+\sigma}\|f - p_k\|_\infty$ where $\sigma = 0$ or 1, then p_k is a minimax polynomial approximant of degree k to f.

11. Show that the set of *natural* cubic splines on a given knot partition $x_0 < x_1 < \cdots < x_n$ is a vector space, V say. Show that V is of dimension $n + 1$.

Why is differentiation *not* a linear transformation from V to V? What is the image of V under the operation of taking the second derivative?

12. Show that one step of Richardson extrapolation applied to the composite Trapezium rule gives the composite Simpson's rule.

Questions involving MATLAB programming

13. Use MATLAB to solve the problem as in Question 4 but with $p \in \Pi_3$: note that the (i,j)th entry of the coefficient matrix is of the form

$$\frac{2^{i+j+2}}{i+j+2}, \quad i,j = 0,1,\ldots,n. \tag{1}$$

Comment on your answer! What will be the best approximation if Π_4 is used?

The $(n+1) \times (n+1)$ coefficient matrix A formed when finding the best fit polynomial $p \in \Pi_n$ to a general function f in the norm corresponding to *this* inner product will always have coefficients of the form (1). Calculate the eigenvalues of A for $n = 2,3,4,5$ (I suggest you just use `eig`): you will note that the smallest eigenvalue gets closer to zero and the largest one grows as n is increased—these matrices are becoming more 'ill-conditioned' as n increases, which is the result of choosing the 'bad basis' $\{1, x, x^2, \ldots, x^n\}$ for Π_n rather than an orthogonal one. This ill-conditioning makes the system of equations difficult to solve; how many accurate decimal places are there in the coefficients of the approximation polynomial to $f(x) = x^3$ for $n = 3,5,10$?

In this question you may want to use `format long` to increase the number of displayed digits.

14. Use `spline` in MATLAB to compute the spline in the question above and then plot it (using `ppval` and `linspace`).

Optional questions

15. For vectors $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$ verify that (i) $\|v\|_\infty = \max_{1 \le i \le n} |v_i|$ and (ii) $\|v\|_1 = \sum_{i=1}^{n} |v_i|$ define norms and draw the 'unit ball' in $\mathbb{R}^2$ (i.e. $\{v \in \mathbb{R}^2 : \|v\| \le 1\}$) in each case.

16. Is

$$\langle f, g \rangle = \int_a^b f(x)g(x) + f'(x)g'(x)\,\mathrm{d}x$$

an inner product on continuously differentiable functions on $[a,b]$?

17. Calculate the best approximation to x on $[0, \pi]$ from $V = \mathrm{span}\{1, \cos x\}$ in the norm associated with the inner product

$$\langle f, g \rangle = \int_0^\pi f(x) g(x) \,\mathrm{d}x.$$

(You might want to consider using calculus).

18. Use `spline` in MATLAB to compute the cubic spline interpolant to $\tanh(10x)$ on $[-5, 5]$ with knots $-5, -4, -3, \ldots, 4, 5$ and end conditions $S'(\pm 5) = 0$. Compare with the lagrange interpolant on the same knots.

19. By considering the identity

$$\int_{x_0}^{x_n} T''(x)^2 \,\mathrm{d}x - \int_{x_0}^{x_n} S''(x)^2 \,\mathrm{d}x$$

$$= \int_{x_0}^{x_n} \big(T''(x) - S''(x)\big)^2 \,\mathrm{d}x + 2 \int_{x_0}^{x_n} \big(T''(x) - S''(x)\big) S''(x) \,\mathrm{d}x$$

prove that amongst all functions $T \in C^2[x_0, x_n]$ which interpolate f at $x_0 < x_1 < \cdots < x_n$, the function which minimises

$$\int_{x_0}^{x_n} T''(x)^2 \,\mathrm{d}x$$

is the natural cubic spline S.

20. On a regular partition of knots $0, 1, 2, \ldots, n$ let S be the interpolating natural cubic spline to $f \in C[a, b]$. (Suppose B_0, B_1, B_{n-1} and B_n are defined to satisfy the natural end conditions and consequently satisfy $B_0(1) = \frac{1}{6} = B_n(n-1)$, $B_1(2) = \frac{1}{4}$, $B_1(0) = 0$, $B_{n-1}(n-2) = \frac{1}{4}$, $B_{n-1}(n) = 0$.) If S is expressed as $\sum_{i=0}^{n} \alpha_i B_i(x)$ in terms of the cubic B-splines show that the coefficients (and hence S) can be determined from solution of the linear system

$$\begin{bmatrix} 1 & 0 & & & & \\ \frac{1}{6} & 1 & \frac{1}{4} & & & \\ & \frac{1}{4} & \ddots & \ddots & & \\ & & \ddots & \ddots & \frac{1}{4} & \\ & & & \frac{1}{4} & \ddots & \frac{1}{6} \\ & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ f(n) \end{bmatrix}.$$

By using Gershgorin's theorem (or otherwise) prove that this matrix is non-singular and deduce that the B-splines $B_0, B_1, B_2, \ldots, B_{n-1}, B_n$ defined here form a basis for the vector space of natural cubic splines.

21. If S is a linear spline interpolant of f on equally spaced points in $[a, b]$ what is $\int_a^b S(x)\,\mathrm{d}x$?

22. Explicitly construct the cubic spline S which interpolates the data $0, \frac{1}{4}, 1, \frac{1}{4}, 0$ at knots $-2, -1, 0, 1, 2$ respectively, and satisfies $S'(\pm 2) = 0$. [Hint: the first 'piece' is $\frac{1}{4}(x+2)^3$ on $[-2, -1]$ and consider symmetry.]

Verify that S'' is continuous at $x = 0$ and check that $S''(0) = -3$

23. Specimen Exam question

Given that n is a non-negative integer, let $\mathcal{P}_n$ denote the set of all polynomials in x of degree n of less. Suppose that f is a function, defined and continuous on the interval $[-1, 1]$ of the real line.

(a) What does it mean to say that $p_n \in \mathcal{P}_n$ is the polynomial of best least-squares approximation for f on $[-1, 1]$ with respect to the norm $\|\cdot\|_2$ defined by

$$\|f\|_2 = \left(\int_{-1}^{1} |f(x)|^2 \, \mathrm{d}x \right)^{1/2} ?$$

(b) Consider the function $f : x \mapsto x^4$ on the interval $[-1, 1]$. By constructing a suitable set of orthogonal polynomials on $[-1, 1]$, or otherwise, find the polynomial $p_3 \in \mathcal{P}_3$ of best least-squares approximation for f on $[-1, 1]$ with respect to the norm $\|\cdot\|_2$, and verify that $p_3(-x) = p_3(x)$ for all $x \in [-1, 1]$.

(c) Now suppose that f is any real-valued function, defined and continuous on the interval $[-1, 1]$ such that $f(-x) = f(x)$ for all $x \in [-1, 1]$. Show that the polynomial $p_n \in \mathcal{P}_n$ of best least-squares approximation for f on the interval $[-1, 1]$ has the property $p_n(-x) = p_n(x)$ for all $x \in [-1, 1]$.

24. (specimen exam question for revision)

Given $n+1$ points $x_0 < x_1 < \cdots < x_n$ and a smooth function f defined on $[x_0, x_n]$, let

$$S_j(x) = \begin{cases} a_j x^3 + b_j x^2 + c_j x + d_j & \text{for } x \in [x_{j-1}, x_j] \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, n$ and

$$S(x) = \begin{cases} \sum_{j=1}^{n} S_j(x) & \text{for } x \in (x_0, x_1) \cup (x_1, x_2) \ \cup \cdots \cup (x_{n-1}, x_n) \\ f(x_i) & \text{for } x = x_i, \ i = 0, 1, \ldots, n. \end{cases}$$

What conditions on the function $S_i(x)$ must be satisfied so that $S(x)$ is a *natural* cubic spline function? (You should assume that these conditions can be satisfied and that the resulting interpolatory natural cubic spline function $S(x)$ is unique.) If $p(x)$ is any linear polynomial, explain why $S(x) + p(x)$ is the interpolatory natural cubic spline for $f(x) + p(x)$. Is this true if $p(x)$ is quadratic?

If $n = 2$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and $f(-1) = \alpha$, $f(0) = 0$, $f(1) = \beta$, explicitly determine the coefficients of $S_1(x)$ and $S_2(x)$ in terms of α and β so that $S(x)$ is the interpolatory natural cubic spline for this data. Show that

$$\int_{-1}^{1} S(x)\,\mathrm{d}x = \frac{3}{8}(\alpha + \beta).$$

For an arbitrary smooth function $g(x)$, let $T(x)$ be the interpolatory natural cubic spline function on the knots $-1, 0, 1$. Using the above, show that the approximation to

$$\int_{-1}^{1} g(x)\,\mathrm{d}x$$

obtained by exactly evaluating

$$\int_{-1}^{1} T(x)\,\mathrm{d}x$$

is the quadrature formula

$$\int_{-1}^{1} g(x)\,\mathrm{d}x \approx \int_{-1}^{1} T(x)\,\mathrm{d}x = \frac{3}{8}g(-1) + \frac{5}{4}g(0) + \frac{3}{8}g(1).$$