Part A Numerical Analysis, Hilary 2020. Problem Sheet 1

1. Construct the Lagrange interpolating polynomial for the data $\begin{array}{c|ccc} x & 0 & 1 & 3 \\ \hline f & 3 & 2 & 6 \end{array}$.

2. If $p_n \in \Pi_n$ interpolates f at $x_0, x_1, \ldots, x_n$, prove that $p_n + q$ is the Lagrange interpolating polynomial to $f + q$ at $x_0, x_1, \ldots, x_n$ whenever $q \in \Pi_n$.

3. Consider interpolating $1/x$ by $p_n \in \Pi_n$ (i.e. at $n+1$ points) on $[1,2]$. If $e(x)$ is the error, show that $|e(x)| \leq 1$ for $x \in [1,2]$ with arbitrarily distributed points, but $|e(x)| \leq 1/2^{(n+1)/2}$ for all $x \in [1,2]$ if $n+1$ is even and half of the interpolation points are in $[1, \frac{3}{2}]$ and half in $(\frac{3}{2}, 2]$. In this latter situation, how many points would be needed to guarantee $|e(x)| \leq 10^{-3}$?

4. Show that $\displaystyle\sum_{k=0}^{n} q(x_k) L_{n,k}(x) = q(x)$ whenever $q \in \Pi_n$. (*Optional:* How many ways can you prove this?) Also, show that $\displaystyle\sum_{k=0}^{n} x_k^l L_{n,k}(x) = x^l$ for nonnegative integers $l \leq n$.

5. Newton–Cotes Quadrature: Find the approximation to the integral

$$\int_0^1 \frac{1}{x+1}\,\mathrm{d}x$$

using the trapezium Rule and Simpson's Rule.

6. Explicitly derive Simpson's Rule from its definition in terms of the quadratic Lagrange interpolating polynomial.

7. Derive the Newton-Cotes quadrature rule of order 3 based on exact integration of the cubic Lagrange interpolating polynomial. [Hint: you may find it helpful to consider symmetries and the substitution $x = x_0 + th$.]

8. Using the Integral Mean Value Theorem, show that

$$\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{2}[f(b) + f(a)] = -\frac{1}{12}(b-a)^3 f''(\eta) \text{ for some } \eta \in (a,b).$$

Hence show that the trapezium rule always overestimates integrals for functions satisfying $f''(x) \geq 0$. Explain geometrically why this is reasonable.

9. Show that Simpson's Rule *exactly* integrates any cubic polynomial on an interval $[a, b]$.

10. If $x_0, x_1, \ldots, x_n$ are distinct real values, then by considering the Lagrange interpolating polynomial in the form $p_n = a_0 + a_1 x + \cdots + a_n x^n$ or otherwise prove that the square matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

is nonsingular.

Questions involving MATLAB programming

11. Using the MATLAB Lagrange interpolation routine that you can download from the course materials web page, compute and plot the Lagrange interpolating polynomial to

(i) The data in Question 1. (You just need to type `lagrange([0,1,3],[3,2,6]`.)

(ii) The UK census data

Year	1951	1961	1971	1981	1991	2001
Pop($\times 10^6$)	48.93	51.38	54.39	54.81	56.20	58.79

To enter this larger set of data, it may be best to create two lists (vectors):

```
years=[1951,1961,1971,1981,1991,2001]
pop=[48.93,51.38,54.39,54.81,56.2,58.79]
```

Plot this data and the interpolating polynomial with "years" on the x-axis.

What do you notice about the value of the interpolating polynomial before 1951 and after 2001? By comparison does the value in 1995 look reasonable?

(iii) The function $f(x) = \frac{1}{1+x^2}$ at equally spaced points on $[-5, 5]$.

Here the command `linspace` is useful: `x=linspace(-5,5,11)` will create a vector `x` with 11 equally spaced values between (and including) -5 and 5. Then `y=1./(1+x.^2)` will create the corresponding values of the function at the points in the vector `x`.

(iv) By changing the view of the plot using either the `axis()` or `ylim()` commands (or otherwise) can you estimate

$$M_{10} := \max_{x \in [-5,5]} |p_{10}(x)|,$$

where $p_n(x)$ is the interpolating polynomial of degree at most n? Vary the number of points at which you interpolate the function and try to record (or compute) M_n for several values of n. What do you think happens as $n \to \infty$? *Optional:* at what rate does this seem to happen (hint: `semilogy()` might be helpful)?

12. Estimate how many equal length intervals $[0, 2]$ should be broken into in order that $f(x)$ be integrated with an accuracy of 10^{-5} using the composite Simpson rule if

$$\max_{x \in [0,2]} |f^{(4)}(x)| = 1.$$

Check how accurate or how pessimistic this estimate is by using the MATLAB function `adaptive_simpson` (available from the course website) for the function $f(x) = \cos(x)$, which you can define in MATLAB with

```
f = @(x) cos(x)
```

You may find `help adaptive_simpson` useful. Compare the numerical quadrature with the exact value of the integral, using `format long` to show more decimal places (`format short` will revert to displaying fewer decimal places).

13. Apply `adaptive_simpson` (see question above) for the following functions:

$$\int_0^{\pi/2} \cos x \,\mathrm{d}x \tag{a}$$

$$\int_{-1}^{1} |x| \,\mathrm{d}x, \qquad \text{(see help abs)} \tag{b}$$

$$\int_{-1}^{3/2} |x| \,\mathrm{d}x \tag{c}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \,\mathrm{d}x \quad \text{approximated by} \quad \int_{-5}^{5} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \,\mathrm{d}x \tag{d}$$

(since $e^{-25/2} \leq 10^{-5}$). [Recall the normal distribution from probability.] You might need to use component-wise exponentiation (`x.^2`) to specify the integrand.

Comment on what you observe in each case, in particular relating what you see to the theory covered in lectures.

Optional questions

14. If you are trying to construct the degree n Lagrange interpolating polynomial to f at $x_0, \ldots, x_n$, can you see (and explain) how the relevant coefficients a_i can be found in the form

$$((\cdots((a_n(x - x_{n-1}) + a_{n-1})(x - x_{n-2}) + a_{n-2}) \cdots + a_2)(x - x_1) + a_1)(x - x_0) + a_0\,?$$

15. Let $w(x) = \prod_{k=0}^{n} (x - x_k)$. Show that the Lagrange interpolating polynomial p_n to f at $x_0, x_1, \ldots, x_n$ can be written as

$$p_n(x) = w(x) \sum_{k=0}^{n} \frac{f(x_k)}{(x - x_k) w'(x_k)}.$$

By dividing by a "clever form of 1", derive the *barycentric formula for Lagrange interpolation*:

$$p_n(x) = \frac{\displaystyle\sum_{k=0}^{n} \frac{w_k}{(x - x_k)} f(x_k)}{\displaystyle\sum_{k=0}^{n} \frac{w_k}{(x - x_k)}},$$

where $w_k = \frac{1}{w'(x_k)}$.

16. In the previous question, what might you expect to go wrong when implementing these formulae on a computer? Despite this, it turns out they are completely well-behaved (numerically stable) although this was only proven recently [Higham, *The numerical stability of barycentric Lagrange interpolation*, 2004], see also the survey article [Berrut & Trefethen, *Barycentric Lagrange interpolation*, 2004].

For real-world code, developed at Oxford, using the barycentric form, see:
`https://github.com/chebfun/chebfun/blob/master/bary.m`

17. Noting that for $b > a$, and any function f continuous on $[a, b]$,

$$\min_{x \in [a,b]} f(x) \le \frac{1}{b - a} \int_a^b f(x)\,\mathrm{d}x \le \max_{x \in [a,b]} f(x),$$

use the Intermediate Value Theorem to show that $\exists \eta \in (a, b)$ satisfying

$$\int_a^b f(x)\,\mathrm{d}x = (b - a) f(\eta).$$

Thus if $G'(x) = g(x) \ge 0$ for $x \in [a, b]$, prove that

$$\int_a^b f(x) g(x)\,\mathrm{d}x = f(\eta) \int_a^b g(x)\,\mathrm{d}x$$

for some $\eta \in (a, b)$. [Note $dG = G'(x)\,\mathrm{d}x$.]

18. *Specimen exam question for revision*

Write down a polynomial $L_{n,k}(x)$ of degree exactly n which satisfies $L_{n,k}(x_i) = 0$ for $i = 0, 1, \ldots, k-1, k+1, \ldots, n$ and $L_{n,k}(x_k) = 1$ where $x_0 < x_1 < \cdots < x_n$. Hence by construction, prove that if data values $f(x_0), f(x_1), \ldots, f(x_n)$ are given then there exists a polynomial P_n of degree at most n satisfying $P_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$. Prove also that P_n is the unique polynomial of degree at most n which satisfies these interpolation conditions.

If $g(x) = \alpha f(x) + \beta$ for some $\alpha, \beta \in \mathbb{R}$ prove that $Q_n(x) = \alpha P_n(x) + \beta$ is the only polynomial of degree at most n which interpolates the function g at the same points x_i, $i = 0, 1, \ldots, n$.

Suppose that $f(x) = f(-x)$ for all x, that n is an odd integer and that the interpolation points are symmetrically placed about the origin so that $x_i = -x_{n-i}$ for $i = 0, 1, \ldots, \frac{n-1}{2}$. For the case $n = 1$, sketch P_1 and show that it is a polynomial of degree zero. Prove by induction for $n = 1, 3, 5, \ldots$ (or otherwise) that P_n is of degree at most $n-1$ and satisfies $P_n(x) = P_n(-x)$. (You may wish to construct a polynomial which includes as one part the $(n-1)^{th}$ degree polynomial $s(x) = (x - x_1) \cdots (x - x_{n-1})$ which satisfies $s(x) = s(-x)$ in your inductive argument.)

Hence show that the polynomial $R_{n+1}(x)$ which interpolates $xf(x)$ at the $n+2$ points $x_0, x_1, \ldots, x_{(n-1)/2}, 0, x_{(n+1)/2}, \ldots, x_{n-1}, x_n$ is in fact of degree at most n.