Part A Numerical Analysis, Hilary 2020. Problem Sheet 2

1. By performing Gauss Elimination (without pivoting), solve

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 24 \\ 25 \end{bmatrix}.$$

From your calculations, write down an LU factorisation of the matrix A above, and verify that $LU = A$. Then by successive back and forwards substitutions (and without further factorisation) solve $Ax = b_2$ where $b_2 = [4\ 7\ 9\ 2]^T$.

2. What is the determinant of the matrix A in the question above? (Note one of the few algebraic properties of the determinant is that $\det(BC) = \det(B)\det(C)$ and you might also want to consider what is the determinant of a triangular matrix).

3. Given an LU factorisation of a matrix A, how might one calculate a column of the inverse of A? Estimate the computational work in calculating A^{-1} and hence in solving $Ax = b$ via explicit computation of A^{-1} and multiplication by b.

Are you now convinced that this is *not* the way to solve linear systems of equations in practice?!

An even worse technique would be to apply GE separately for each column: what would the computational cost be then?

4. Suppose A is a real $n \times n$ matrix with $n \geq 2$ and that the permutation matrix

$$P = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & . & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

Show that premultiplication of A by P reverses the order of the rows of A.

If $A = LU$ is an LU factorisation of A (without pivoting), what is the structure of PLP? Hence describe how to calculate a factorisation $A = \hat{U}\hat{L}$ where $\hat{U}$ is unit upper triangular and $\hat{L}$ is lower triangular.

5. If $\|x\| := \sqrt{x^T x}$ is the usual (Euclidean) length of a vector $x \in \Re^n$, show that the vector Qx has the same length whenever Q is an orthogonal $n \times n$ matrix.

If we define the angle between vectors $x, y \in \Re^n$ as

$$\angle(x, y) := \cos^{-1}\left(\frac{x^T y}{\|x\|\|y\|}\right)$$

show that the angle between Qx and Qy is unchanged.

6. Suppose that A is a square nonsingular matrix. Prove that the factors Q and R featuring in the QR factorisation of A are unique if the diagonal entries of R are all positive. How many possibilities are there if this restriction is removed?

7. By considering the QR factorisation in which the diagonal entries of R are all positive as in the question above (or otherwise), prove that any orthogonal matrix may be expressed as the product of Householder matrices.

8. Determine the eigenvalues of a Householder matrix.

9. Show that if $x \in \Re^n$ then (denoting by $J(i, j)$ a Givens rotation)

$$J(1, n)J(1, n-1)\cdots J(1, 3)J(1, 2)\, x = \begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some β and further prove that $\beta^2 = x^T x$.

10. Describe a sequence of $(n-1) + (n-2) + \cdots + 2 + 1 = n(n-1)/2$ Givens matrices which when applied sequentially (as premultiplications) will reduce an $n \times n$ matrix to upper triangular form. (Hint: see question above). You have thus established another algorithm for QR factorisation. What is Q?

11. First, let A, B be $n \times n$ square matrices. Prove that the eigenvalues of AB and BA are the same. (you can first assume B is nonsingular).

(Hard but recommended): Next let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$. Prove that the nonzero eigenvalues of AB and BA are again the same.

Questions involving MATLAB programming

12. In MATLAB the 'backslash', \, solves linear systems of equations via Gauss Elimination with partial pivoting: thus `x=A\b` will perform a permuted LU factorisation (sometimes called a PLU factorisation: $PA = LU$) of A and solves $Ly = Pb$ via forward substitution and then $Ux = y$ via back substitution, giving back just the solution vector x.

Verify your solutions to the two linear systems in question 1 above.

13. Perform an LU factorisation with partial pivoting on the matrix

$$A = \begin{bmatrix} 2/3 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3/2 & 4 \end{bmatrix}$$

and write down the permutation matrix P, the unit lower triangular matrix of multipliers and the upper triangular matrix U such that $PA = LU$ which you should verify.

Check your answer with MATLAB:
`[L,U,P]=lu(A)`
will compute P, L and U for you - see how much easier! (but you must do this by hand once in your life and if you've correctly done the above then join the club! else look and see why you've gone wrong).

14. Note that `A = randn(7, 7);` creates a random matrix in MATLAB with entries taken from a normal distribution with mean zero and variance 1. Also, you can time a command using `tic` and `toc` as in `tic; [L, U, P] = lu(A); toc`

Using the above, for random matrices of dimensions 2^k for $k = 5, 6, \ldots, 10$, see by what factor the time to compute an LU factorisation grows as you double the dimension. (I wouldn't try for too much larger dimension else you might be waiting a long time! - if you do want to interrupt, then control and c keys pressed together should stop the current computation and return you to a MATLAB prompt).

You might find a `for` loop useful here. Make sure you do not time the creation of the matrix, i.e., keep the `randn` command outside the `tic toc`.

15. Continuing the above question, using the timing for the matrix of dimension $2^{10} = 1024 \approx 10^3$, estimate how many millions of floating point operations (flops) the computer you are working on can compute in 1 second. Hence estimate the dimension of a matrix for which LU factorisation (or equivalently Gaussian elimination) would take 1 year on your computer (if it had enough memory to store all of the numbers).

16. If $A = QR$ is a QR factorisation of A, show that this provides a special type of triangular factorisation of $A^T A$ in which $A^T A$ is expressed as the product of an upper triangular matrix followed by its transpose. (Such decomposition $B = R^T R$ is called the **Cholesky factorization** for a positive definite matrix).

In MATLAB, `[Q, R] = qr(A)` calculates a QR factorisation of a matrix and `Q*Q'` will test if a matrix is orthogonal. Verify numerically the existence of the special triangular factorisation above for any matrix (you may want to use `randn` to construct A).

17. Investigate the use of QR factorisation in the solution of linear systems of equations: generate a random square matrix, a right hand side vector of ones and the solution using LU factorisation and forward and back substitution using something like

```
A = randn(8, 8); b = ones(8, 1); x = A\b
```

and compare x to the solution y, say using QR factorisation:

```
[Q, R] = qr(A)
y = R\(Q'*b)
```

Note here that \ used with a triangular matrix should perform the appropriate (forwards or backwards) substitution whereas for a general matrix it performs PLU factorisation. You may want to use `format long` to see more digits in x and y since they should be very similar!

18. Using a loop and `tic` and `toc` compare the time it takes to do PLU and QR factorisations. For example, for random matrices of dimension 2^5 to 2^{10}

```
for k=5:10, A=randn(2^k); tic, [L,U,P]=lu(A); toc,...
tic, [Q,R]=qr(A); toc, end
```

should give some timings. (Note how you use dots in MATLAB to continue onto the next line). What do you think the computational work is for QR factorisation given that LU is to leading order $\frac{2}{3}n^3$? Note `qr` uses Householder matrices as described in lectures to compute the QR factorisation.

Optional questions

19. Prove that the product of two lower triangular matrices is lower triangular and that the inverse of a non-singular lower triangular matrix is lower triangular. Deduce similar results for upper triangular matrices.

20. Show that if $x, y \in \Re^n$ with $x \neq 0, y \neq 0$ then the outer product matrix, xy^T has an $n-1$–dimensional kernel. Identify the Image (also called the Range) of this matrix. Hence, identify an eigenvector which corresponds to a generally non-zero eigenvalue and give the condition under which this eigenvalue is also zero.

21. If S is a real skew-symmetric matrix, so that $S^T = -S$, and assuming that $I - S$ is nonsingular, show that $(I - S)^{-1}(I + S)$ is an orthogonal matrix. (You may want to convince yourself that for a nonsingular matrix $(A^{-1})^T = (A^T)^{-1}$.)

22. If $x \in \Re^n$ show that postmultiplication of the row vector x^T by $J(i, j, \theta)$ with an appropriately chosen value of θ which you should give, will make the j^{th} entry of the resulting row vector equal to zero.

23. Using the results of the above questions can you show directly (and constructively) that any matrix $A \in \Re^{n\times n}$ admits a factorisation LQ with L a lower triangular matrix and Q orthogonal. (Note if $B = QR$ then $B^T = R^T Q^T$ certainly is another way of doing this).

24. If A is a real matrix with m rows and n columns (i.e. a rectangular matrix when $m \neq n$), convince yourself that the method described in lectures for QR factorisation using Householder matrices could equally well be applied for $m \neq n$. How many Householder matrices are required if $m > n$ and how many if $m \leq n$?

25. Specimen exam question for revision

Usually GE is applied to square matrices. Suppose, however, that we apply GE (without pivoting) to a matrix with m rows and n columns with $m > n$. Assume that no zeros on the diagonal are encountered.

(a) Describe the elimination process algorithmically in the style of a computer program, explaining exactly what loops and arithmetic operations are involved in the elimination process.

(b) Describe the matrix factorization that has been accomplished by GE. How have we decomposed A? Be sure to be explicit about the dimensions of matrices involved, zeros vs. nonzero entries, etc.

(c) Let b be a column vector of dimension m. Depending on b, the system of equations $Ax = b$ may or may not have a solution. Using the results of the Gaussian elimination, show how to compute a $n-$vector x that is the solution, if it exists. Give a condition on x in terms of the results of Gaussian elimination that determines whether or not $Ax = b$ has a solution.

(d) Suppose now that a zero on the diagonal is encountered after all in the elimination process, at step k. Show that the upper left $k \times k$ block of A is singular.

26. Specimen exam question for revision

(a) Define what it means for two n-vectors x and y to be orthogonal, and what it means for an $n \times n$ matrix Q to be orthogonal.

(b) Suppose an $n \times n$ matrix A has LU and QR factorisations

$$A = LU, \qquad A = QR$$

in the usual senses of these terms. State precisely what forms the matrices L, U, Q and R have in these factorisations.

(c) Name (but do not describe) the standard algorithms for computing LU and QR factorisations.

(d) Suppose A is nonsingular. Show that this implies that the diagonal entries of U and of R are all nonzero.

(e) Let $q_1, \ldots, q_n$ denote the columns of Q, and let $\ell_1, \ldots, \ell_n$ denote the columns of L. For any k with $1 \leq k \leq n$, let $\mathcal{Q}_k$ and $\mathcal{L}_k$ denote the k-dimensional subspaces

$$\mathcal{Q}_k = \texttt{span}\{q_1, \ldots, q_k\}, \qquad \mathcal{L}_k = \texttt{span}\{\ell_1, \ldots, \ell_k\},$$

i.e. the subspaces of all linear combinations of the indicated k vectors. Still assuming A is nonsingular, show that for each k,

$$\mathcal{Q}_k = \mathcal{L}_k.$$

(f) Let m denote the last row of L^{-1}. Still assuming A is nonsingular, show that

$$m = Cq_n^T$$

for some constant C. What is this constant?