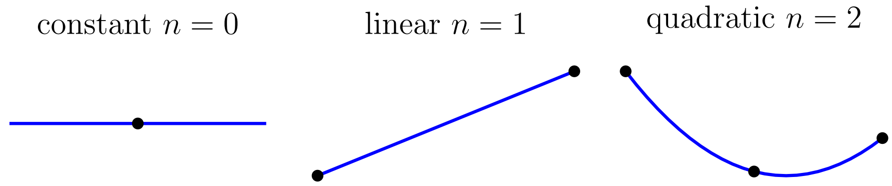
Numerical Analysis Hilary Term 2020
Lecture 1: Lagrange Interpolation

These lecture notes are adapted from the numerical analysis textbook by Süli and Mayers. This first lecture comes from Chapter 6 of the book.

Notation: $\Pi_n = \{\text{real polynomials of degree} \le n\}$

Setup: Given data f_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p_n such that $p_n(x_i) = f_i$? Such a polynomial is said to **interpolate** the data, and (as we shall see) can approximate f at other values of x if f is smooth enough. This is the most basic question in approximation theory.

E.g.:



Theorem. $\exists p_n \in \Pi_n$ such that $p_n(x_i) = f_i$ for $i = 0, 1, \ldots, n$.

Proof. Consider, for $k = 0, 1, \ldots, n$, the "cardinal polynomial"

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \in \Pi_n. \tag{1}$$

Then $L_{n,k}(x_i) = \delta_{ik}$, that is,

$$L_{n,k}(x_i) = 0 \text{ for } i = 0, \ldots, k-1, k+1, \ldots, n \text{ and } L_{n,k}(x_k) = 1.$$

So now define

$$p_n(x) = \sum_{k=0}^{n} f_k L_{n,k}(x) \in \Pi_n \tag{2}$$

$\implies$

$$p_n(x_i) = \sum_{k=0}^{n} f_k L_{n,k}(x_i) = f_i \text{ for } i = 0, 1, \ldots, n. \qquad \square$$

The polynomial (2) is the **Lagrange interpolating polynomial**.

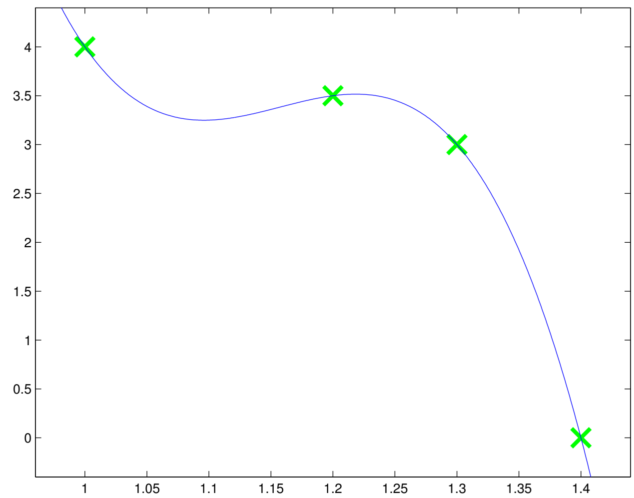
Theorem. The interpolating polynomial of degree $\le n$ is unique.

Proof. Consider two interpolating polynomials $p_n, q_n \in \Pi_n$. Their difference $d_n = p_n - q_n \in \Pi_n$ satisfies $d_n(x_k) = 0$ for $k = 0, 1, \ldots, n$. i.e., d_n is a polynomial of degree at most n but has at least $n + 1$ distinct roots. Algebra $\implies$ $d_n \equiv 0 \implies p_n = q_n$. $\square$

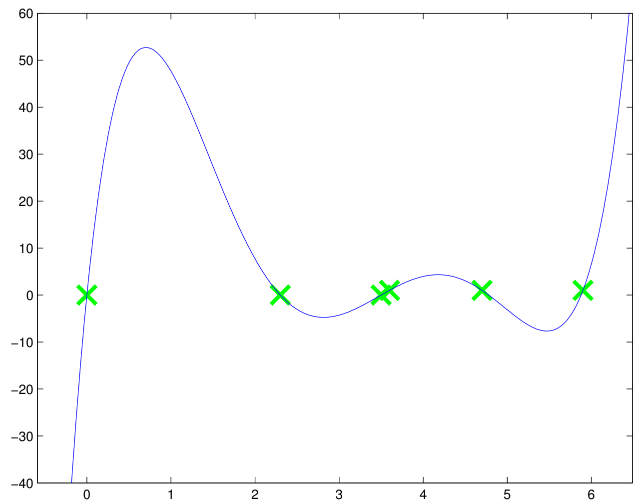
Matlab:

```
>> help lagrange
 LAGRANGE  Plots the Lagrange polynomial interpolant for the
 given DATA at the given KNOTS

>> lagrange([1,1.2,1.3,1.4],[4,3.5,3,0]);
```




```
>> lagrange([0,2.3,3.5,3.6,4.7,5.9],[0,0,0,1,1,1]);
```



Data from an underlying smooth function: Suppose that $f(x)$ has at least $n+1$ smooth derivatives in the interval (x_0, x_n). Let $f_k = f(x_k)$ for $k = 0, 1, \ldots, n$, and let p_n be the Lagrange interpolating polynomial for the data (x_k, f_k), $k = 0, 1, \ldots, n$.

Error: How large can the error $f(x) - p_n(x)$ be on the interval $[x_0, x_n]$?

Theorem. For every $x \in [x_0, x_n]$ there exists $\xi = \xi(x) \in (x_0, x_n)$ such that

$$e(x) \stackrel{\text{def}}{=} f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}, \tag{3}$$

where $f^{(n+1)}$ is the $(n+1)$-st derivative of f.

Proof. Trivial for $x = x_k$, $k = 0, 1, \ldots, n$ as $e(x) = 0$ by construction. So suppose $x \neq x_k$. Let

$$\phi(t) \stackrel{\text{def}}{=} e(t) - \frac{e(x)}{\pi(x)}\pi(t),$$

where

$$\begin{aligned} \pi(t) &\stackrel{\text{def}}{=} (t - x_0)(t - x_1) \cdots (t - x_n) \\ &= t^{n+1} - \left(\sum_{i=0}^{n} x_i\right) t^n + \cdots (-1)^{n+1} x_0 x_1 \cdots x_n \\ &\in \Pi_{n+1}. \end{aligned}$$

Now note that ϕ vanishes at $n + 2$ points x and x_k, $k = 0, 1, \ldots, n$. $\Longrightarrow$ ϕ' vanishes at $n + 1$ points $\xi_0, \ldots, \xi_n$ between these points $\Longrightarrow$ ϕ'' vanishes at n points between these new points, and so on until $\phi^{(n+1)}$ vanishes at an (unknown) point ξ in (x_0, x_n). But

$$\phi^{(n+1)}(t) = e^{(n+1)}(t) - \frac{e(x)}{\pi(x)}\pi^{(n+1)}(t) = f^{(n+1)}(t) - \frac{e(x)}{\pi(x)}(n+1)!$$

since $p_n^{(n+1)}(t) \equiv 0$ and because $\pi(t)$ is a monic polynomial of degree $n+1$. The result then follows immediately from this identity since $\phi^{(n+1)}(\xi) = 0$.

$\square$

Example: $f(x) = \log(1 + x)$ on $[0, 1]$. Here, $|f^{(n+1)}(\xi)| = n!/(1+\xi)^{n+1} < n!$ on $(0, 1)$. So $|e(x)| < |\pi(x)|n!/(n + 1)! \leq 1/(n + 1)$ since $|x - x_k| \leq 1$ for each x, x_k, $k = 0, 1, \ldots, n$, in $[0, 1] \Longrightarrow |\pi(x)| \leq 1$. This is probably pessimistic for many x, e.g. for $x = \frac{1}{2}$, $\pi(\frac{1}{2}) \leq 2^{-(n+1)}$ as $|\frac{1}{2} - x_k| \leq \frac{1}{2}$.

This shows the important fact that the error can be large at the end points when samples $\{x_k\}$ are equispaced points, an effect known as the "Runge phenomena" (Carl Runge, 1901). There is a famous example due to Runge, where the error from the interpolating polynomial approximation to $f(x) = (1 + x^2)^{-1}$ for $n + 1$ equally-spaced points on $[-5, 5]$ diverges near ± 5 as n tends to infinity: try this example with `lagrange` from the website in Matlab[1]

Building Lagrange interpolating polynomials from lower degree ones.

Notation: Let $Q_{i,j}$ be the Lagrange interpolating polynomial at x_k, $k = i, \ldots, j$.

Theorem.

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{4}$$

Proof. Let $s(x)$ denote the right-hand side of (4). Because of uniqueness, we simply wish to show that $s(x_k) = f_k$. For $k = i + 1, \ldots, j - 1$, $Q_{i+1,j}(x_k) = f_k = Q_{i,j-1}(x_k)$, and hence

$$s(x_k) = \frac{(x_k - x_i)Q_{i+1,j}(x_k) - (x_k - x_j)Q_{i,j-1}(x_k)}{x_j - x_i} = f_k.$$

[1] There is a beautiful solution to this issue, Chebyshev interpolation: choose $\{x_k\}$ cleverly, essentially to minimise $\max_{x\in[x_0,x_n]} |(x - x_0)(x - x_1) \cdots (x - x_n)|$ in (3). This results in taking more points near the endpoints. See Trefethen's book Approximation Theory and Approximation Practices, SIAM.

We also have that $Q_{i+1,j}(x_j) = f_j$ and $Q_{i,j-1}(x_i) = f_i$, and hence

$$s(x_i) = Q_{i,j-1}(x_i) = f_i \text{ and } s(x_j) = Q_{i+1,j}(x_j) = f_j. \qquad \square$$

Comment: This can be used as the basis for constructing interpolating polynomials. In books: may find topics such as the Newton form and divided differences.

Generalisation: Given data f_i and g_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p such that $p(x_i) = f_i$ and $p'(x_i) = g_i$? (i.e., interpolate derivatives in addition to values)

Theorem. There is a unique polynomial $p_{2n+1} \in \Pi_{2n+1}$ such that $p_{2n+1}(x_i) = f_i$ and $p'_{2n+1}(x_i) = g_i$ for $i = 0, 1, \ldots, n$.

Construction: Given $L_{n,k}(x)$ in (1), let

$$\begin{aligned} H_{n,k}(x) = &\ [L_{n,k}(x)]^2(1 - 2(x - x_k)L'_{n,k}(x_k)) \\ \text{and } K_{n,k}(x) = &\ [L_{n,k}(x)]^2(x - x_k). \end{aligned}$$

Then

$$p_{2n+1}(x) = \sum_{k=0}^{n}[f_k H_{n,k}(x) + g_k K_{n,k}(x)] \tag{5}$$

interpolates the data as required. The polynomial (5) is called the **Hermite interpolating polynomial**. Note that $H_{n,k}(x_i) = \delta_{ik}$ and $H'_{n,k}(x_i) = 0$, and $K_{n,k}(x_i) = 0$, $K'_{n,k}(x_i) = \delta_{ik}$.

Theorem. Let p_{2n+1} be the Hermite interpolating polynomial in the case where $f_i = f(x_i)$ and $g_i = f'(x_i)$ and f has at least $2n+2$ smooth derivatives. Then, for every $x \in [x_0, x_n]$,

$$f(x) - p_{2n+1}(x) = [(x - x_0)(x - x_1)\cdots(x - x_n)]^2 \frac{f^{(2n+2)}(\xi)}{(2n+2)!},$$

where $\xi \in (x_0, x_n)$ and $f^{(2n+2)}$ is the $(2n+2)$nd derivative of f.

Proof (non-examinable): see Süli and Mayers, Theorem 6.4. $\square$

We note that as $x_k \to 0$ in (3), we essentialy recover Taylor's theorem with $p_n(x)$ equal to the first $n+1$ terms in Taylor's expansion. Taylor's theorem can be regarded as a special case of Lagrange interpolation where we interpolate high-order derivatives at a single point.