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Numerical Analysis Hilary Term 2020  
Lecture 10: Best Approximation in Inner-Product Spaces

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**Best approximation of functions:** given a function  $f$  on  $[a, b]$ , find the “closest” polynomial/piecewise polynomial (see later sections)/ trigonometric polynomial (truncated Fourier series).

**Norms:** are used to measure the size of/distance between elements of a vector space. Given a vector space  $V$  over the field  $\mathbb{R}$  of real numbers, the mapping  $\|\cdot\| : V \rightarrow \mathbb{R}$  is a **norm** on  $V$  if it satisfies the following axioms:

- (i)  $\|f\| \geq 0$  for all  $f \in V$ , with  $\|f\| = 0$  if, and only if,  $f = 0 \in V$ ;
- (ii)  $\|\lambda f\| = |\lambda| \|f\|$  for all  $\lambda \in \mathbb{R}$  and all  $f \in V$ ; and
- (iii)  $\|f + g\| \leq \|f\| + \|g\|$  for all  $f, g \in V$  (the **triangle inequality**).

**Examples:** 1. For vectors  $x \in \mathbb{R}^n$ , with  $x = (x_1, x_2, \dots, x_n)^T$ ,

$$\|x\| \equiv \|x\|_2 = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}} = \sqrt{x^T x}$$

is the  $\ell^2$ - or vector two-norm.

2. For continuous functions on  $[a, b]$ ,

$$\|f\| \equiv \|f\|_\infty = \max_{x \in [a, b]} |f(x)|$$

is the  $L^\infty$ - or  $\infty$ -norm.

3. For integrable functions on  $(a, b)$ ,

$$\|f\| \equiv \|f\|_1 = \int_a^b |f(x)| dx$$

is the  $L^1$ - or one-norm.

4. For functions in

$$V = L_w^2(a, b) \equiv \{f : [a, b] \rightarrow \mathbb{R} \mid \int_a^b w(x)[f(x)]^2 dx < \infty\}$$

for some given **weight** function  $w(x) > 0$  (this certainly includes continuous functions on  $[a, b]$ , and piecewise continuous functions on  $[a, b]$  with a finite number of jump-discontinuities),

$$\|f\| \equiv \|f\|_2 = \left( \int_a^b w(x)[f(x)]^2 dx \right)^{\frac{1}{2}}$$

is the  $L^2$ - or two-norm—the space  $L^2(a, b)$  is a common abbreviation for  $L_w^2(a, b)$  for the case  $w(x) \equiv 1$ .

**Note:**  $\|f\|_2 = 0 \implies f = 0$  almost everywhere on  $[a, b]$ . We say that a certain property  $P$  holds *almost everywhere* (a.e.) on  $[a, b]$  if property  $P$  holds at each point of  $[a, b]$  except perhaps on a subset  $S \subset [a, b]$  of zero measure. We say that a set  $S \subset \mathbb{R}$  has *zero measure* (or that it is of *measure zero*) if for any  $\varepsilon > 0$  there exists a sequence  $\{(\alpha_i, \beta_i)\}_{i=1}^\infty$  of subintervals of  $\mathbb{R}$  such that

$S \subset \cup_{i=1}^{\infty} (\alpha_i, \beta_i)$  and  $\sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \varepsilon$ . Trivially, the empty set  $\emptyset (\subset \mathbb{R})$  has zero measure. Any finite subset of  $\mathbb{R}$  has zero measure. Any countable subset of  $\mathbb{R}$ , such as the set of all natural numbers  $\mathbb{N}$ , the set of all integers  $\mathbb{Z}$ , or the set of all rational numbers  $\mathbb{Q}$ , is of measure zero.

**Least-squares polynomial approximation:** aim to find the best polynomial approximation to  $f \in L_w^2(a, b)$ , i.e., find  $p_n \in \Pi_n$  for which

$$\|f - p_n\|_2 \leq \|f - q\|_2 \quad \forall q \in \Pi_n.$$

Seeking  $p_n$  in the form  $p_n(x) = \sum_{k=0}^n \alpha_k x^k$  then results in the minimization problem

$$\min_{(\alpha_0, \dots, \alpha_n)} \int_a^b w(x) \left[ f(x) - \sum_{k=0}^n \alpha_k x^k \right]^2 dx.$$

The unique minimizer can be found from the (linear) system

$$\frac{\partial}{\partial \alpha_j} \int_a^b w(x) \left[ f(x) - \sum_{k=0}^n \alpha_k x^k \right]^2 dx = 0 \quad \text{for each } j = 0, 1, \dots, n,$$

but there is important additional structure here.

**Inner-product spaces:** a real **inner-product space** is a vector space  $V$  over  $\mathbb{R}$  with a mapping  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  (the **inner product**) for which

- (i)  $\langle v, v \rangle \geq 0$  for all  $v \in V$  and  $\langle v, v \rangle = 0$  if, and only if  $v = 0$ ;
- (ii)  $\langle u, v \rangle = \langle v, u \rangle$  for all  $u, v \in V$ ; and
- (iii)  $\langle \alpha u + \beta v, z \rangle = \alpha \langle u, z \rangle + \beta \langle v, z \rangle$  for all  $u, v, z \in V$  and all  $\alpha, \beta \in \mathbb{R}$ .

**Examples:** 1.  $V = \mathbb{R}^n$ ,

$$\langle x, y \rangle = x^T y = \sum_{i=1}^n x_i y_i,$$

where  $x = (x_1, \dots, x_n)^T$  and  $y = (y_1, \dots, y_n)^T$ .

2.  $V = L_w^2(a, b) = \{f : (a, b) \rightarrow \mathbb{R} \mid \int_a^b w(x)[f(x)]^2 dx < \infty\}$ ,

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx,$$

where  $f, g \in L_w^2(a, b)$  and  $w$  is a weight-function, defined, positive and integrable on  $(a, b)$ .

**Notes:** 1. Suppose that  $V$  is an inner product space, with inner product  $\langle \cdot, \cdot \rangle$ . Then  $\langle v, v \rangle^{\frac{1}{2}}$  defines a norm on  $V$  (see the final paragraph on the last page for a proof). In Example 2 above, the norm defined by the inner product is the (weighted)  $L^2$ -norm.

2. Suppose that  $V$  is an inner product space, with inner product  $\langle \cdot, \cdot \rangle$ , and let  $\|\cdot\|$  denote the norm defined by the inner product via  $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$ , for  $v \in V$ . The angle  $\theta$  between  $u, v \in V$  is

$$\theta = \cos^{-1} \left( \frac{\langle u, v \rangle}{\|u\| \|v\|} \right).$$

Thus  $u$  and  $v$  are orthogonal in  $V \iff \langle u, v \rangle = 0$ .

E.g.,  $x^2$  and  $\frac{3}{4} - x$  are orthogonal in  $L^2(0, 1)$  with inner product  $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$  as

$$\int_0^1 x^2 \left(\frac{3}{4} - x\right) dx = \frac{1}{4} - \frac{1}{4} = 0.$$

**3. Pythagoras Theorem:** Suppose that  $V$  is an inner-product space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$  defined by this inner product. For any  $u, v \in V$  such that  $\langle u, v \rangle = 0$  we have

$$\|u \pm v\|^2 = \|u\|^2 + \|v\|^2.$$

**Proof.**

$$\begin{aligned} \|u \pm v\|^2 &= \langle u \pm v, u \pm v \rangle = \langle u, u \pm v \rangle \pm \langle v, u \pm v \rangle && \text{[axiom (iii)]} \\ &= \langle u, u \pm v \rangle \pm \langle u \pm v, v \rangle && \text{[axiom (ii)]} \\ &= \langle u, u \rangle \pm \langle u, v \rangle \pm \langle u, v \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle && \text{[orthogonality]} \\ &= \|u\|^2 + \|v\|^2. \end{aligned}$$

**4. The Cauchy–Schwarz inequality:** Suppose that  $V$  is an inner-product space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$  defined by this inner product. For any  $u, v \in V$ ,

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

**Proof.** For every  $\lambda \in \mathbb{R}$ ,

$$0 \leq \langle u - \lambda v, u - \lambda v \rangle = \|u\|^2 - 2\lambda \langle u, v \rangle + \lambda^2 \|v\|^2 = \phi(\lambda),$$

which is a quadratic in  $\lambda$ . The minimizer of  $\phi$  is at  $\lambda_* = \langle u, v \rangle / \|v\|^2$ , and thus since  $\phi(\lambda_*) \geq 0$ ,  $\|u\|^2 - \langle u, v \rangle^2 / \|v\|^2 \geq 0$ , which gives the required inequality.  $\square$

**5. The triangle inequality:** Suppose that  $V$  is an inner-product space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$  defined by this inner product. For any  $u, v \in V$ ,

$$\|u + v\| \leq \|u\| + \|v\|.$$

**Proof.** Note that

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2.$$

Hence, by the Cauchy–Schwarz inequality,

$$\|u + v\|^2 \leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2.$$

Taking square-roots yields

$$\|u + v\| \leq \|u\| + \|v\|.$$

$\square$

**Note:** The function  $\| \cdot \| : V \rightarrow \mathbb{R}$  defined by  $\|v\| := \langle v, v \rangle^{\frac{1}{2}}$  on the inner-product space  $V$ , with inner product  $\langle \cdot, \cdot \rangle$ , trivially satisfies the first two axioms of norm on  $V$ ; this is a consequence of  $\langle \cdot, \cdot \rangle$  being an inner product on  $V$ . Result 5 above implies that  $\| \cdot \|$  also satisfies the third axiom of norm, the triangle inequality.