1 Second-order linear boundary value problems

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1.1 Basic notation and concepts

In this section, we will develop various techniques to analyse and solve ordinary differential equations (ODEs), in particular *inhomogeneous linear boundary value problems* (BVPs). We start by briefly explaining what is meant by each piece of this expression. Although everything to follow can in principle be generalised to ODEs of arbitrary order, we restrict our attention to second order ODEs for the moment.

A second-order *linear* ODE is an equation of the form

$$\mathfrak{L}y(x) = f(x), \tag{1.1}$$

where f is a given *forcing function* and $\mathfrak{L}$ is a *linear differential operator*, that is,

$$\mathfrak{L}y(x) \equiv P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) \tag{1.2a}$$

$$\equiv P_2(x)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P_1(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + P_0(x)y(x), \tag{1.2b}$$

for some given coefficients $P_0(x)$, $P_1(x)$, $P_2(x)$. The operator $\mathfrak{L}$ is linear in the sense that

$$\mathfrak{L}\big[\alpha_1 y_1(x) + \alpha_2 y_2(x)\big] \equiv \alpha_1\mathfrak{L}y_1(x) + \alpha_2\mathfrak{L}y_2(x), \tag{1.3}$$

for any constants α_i and functions $y_i(x)$. Here, and henceforth unless explicitly stated otherwise, we assume that y is sufficiently smooth for all the required derivatives to exist and be continuous. We will also assume that the coefficents P_i are at least continuous and (for reasons that will become clear) that P_2 is nonzero in the range of x of interest.

The linear ODE (1.1) is said to be *homogeneous* if the right-hand side f is identically zero, and if not then the equation is *inhomogeneous*. We will refer frequently to the homogeneous and inhomogeneous (or "Non-homogeneous") versions of (1.1), which we label as follows:

$$\textbf{homogeneous:} \qquad \mathfrak{L}y = 0, \tag{H}$$

$$\textbf{inhomogeneous:} \qquad \mathfrak{L}y = f \not\equiv 0. \tag{N}$$

Generally, we expect to need to supplement a second-order ODE of the form (1.1) with *two* boundary conditions to get a unique solution for $y(x)$, and the term *boundary value problem* refers to the way in which those boundary conditions are imposed. Much of the Differential Equations I course concerns the solution of *initial value problems* (IVPs), where the "initial values" of y and y' are given at a single point $x = a$, say. In a BVP, the ODE (1.1) is posed on an interval, say $a < x < b$, and the boundary conditions involve the values of y and y' at both ends of the domain $x = a$ and $x = b$. Provided the coefficients $P_i(x)$ and the forcing function $f(x)$ are sufficiently well behaved (and $P_2(x) \neq 0$), Picard's Theorem guarantees that an IVP for a linear ODE of the form (1.1) has a unique solution in a neighbourhood of the initial point $x = a$, but we will see that the same cannot be said of a linear BVP.

Example 1.1. Second order IVP and BVP

The simple 2nd order linear inhomogeneous ODE

$$y'' + y = 1 \tag{1.4}$$

has the general solution $y(x) = 1 + c_1 \cos x + c_2 \sin x$, *where* c_1 *and* c_2 *are arbitrary* integration constants. *A typical IVP would involve solving* (1.4) *in* $x > 0$ *subject to the initial conditions* $y(0) = 1$ *and* $y'(0) = 2$. *By imposing the two initial conditions, we can easily solve for the integration constants and thus obtain the solution* $y(x) = 1 + 2\sin x$.

A typical BVP would be to solve (1.4) *on an interval, say* $0 < x < \pi$, *subject to the boundary conditions* $y(0) = 1$ *and* $y'(\pi) = 2$. *Again, we can solve for the arbitrary constants, and this time we obtain the solution* $y(x) = 1 - 2\sin x$.

Suppose we replace the right-hand side of (1.4) *with a more complicated forcing function, for example*

$$y''(x) + y(x) = \tan x. \tag{1.5}$$

In principle, this ODE is solvable, subject to suitable boundary conditions, but now it is not at all obvious how to "spot" the particular integral!

Finally, suppose we slightly alter the boundary conditions to $y(0) = 1$ *and* $y(\pi) = 2$. *One can easily confirm that the ODE* (1.4) *has* no solution *subject to the modified boundary conditions.*

In the remainder of this section, we will derive general methods to solve ODEs of the form (1.1), as well as addressing the following general questions.

1. How can we construct a particular integral for the ODE (1.1) for arbitrary forcing function f?
2. Given suitable boundary conditions, when does a solution exist? When is it unique?

1.2 Space of solutions

If we ignore boundary conditions for the moment, then the following properties of solutions of (H) and (N) are easily established.

(i) The solutions of (H) form a vector space since, if $\mathfrak{L}y_1 = 0 = \mathfrak{L}y_2$, then $\mathfrak{L}[\alpha y_1 + \beta y_2] = 0$.

(ii) If y and Y satisfy (N), then $y - Y$ satisfies (H).

(iii) It follows that the general solution of (N) may be written in the form

$$y(x) = \underbrace{y_{\mathrm{PI}}(x)}_{\text{any solution of (N)}} + \underbrace{y_{\mathrm{CF}}(x)}_{\text{general solution of (H)}} \tag{1.6}$$

where y_{PI} is called the *particular integral* and y_{CF} the *complementary function.*

(iv) For a second-order ODE, the vector space of solutions to (H) has dimension two (see below). The complementary function therefore takes the form

$$y_{\mathrm{CF}}(x) = c_1 y_1(x) + c_2 y_2(x), \tag{1.7}$$

where c_1, c_2 are arbitrary constants, and y_1, y_2 are any two *linearly independent* solutions to (H).

1.3 Linear independence; the Wronskian

A pair of functions $y_1(x)$, $y_2(x)$ is *linearly independent* if there is no non-trivial linear combination that vanishes identically; in other words if

$$c_1 y_1(x) + c_2 y_2(x) \equiv 0 \quad \Leftrightarrow \quad c_1 = c_2 = 0. \tag{1.8}$$

They are *linearly dependent* if c_i, not both zero, can be found such that $c_1 y_1(x) + c_2 y_2(x)$ is identically zero. Provided y_1, y_2 are differentiable, this would also entail $c_1 y_1'(x) + c_2 y_2'(x) \equiv 0$. Therefore

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \equiv \mathbf{0}, \tag{1.9}$$

and non-trivial solutions can exist for (c_1, c_2) if and only if the determinant of the matrix is zero.

We define the *Wronskian* of a pair of functions to be this determinant:

$$W(x) = W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1(x) y_2'(x) - y_2(x) y_1'(x). \tag{1.10}$$

From what we have just seen, we conclude the following.

Proposition 1.1. *If two functions are linearly dependent then their Wronskian vanishes.*

The converse to this statement is not necessarily true, however. For example, the following (once) differentiable functions:

$$y_1(x) = \begin{cases} 0 & x < 0, \\ x^2 & x \geq 0, \end{cases} \qquad y_2(x) = \begin{cases} x^2 & x < 0, \\ 0 & x \geq 0, \end{cases} \tag{1.11}$$

are easily shown to be linearly independent, but have Wronskian equal to zero [**exercise**]. We will now show that there *is* a partial converse to Proposition 1.1 for the case where y_1 and y_2 are solutions to (H).

Suppose that y_1 and y_2 are two solutions to (H), i.e.

$$P_2 y_1'' + P_1 y_1' + P_0 y_1 = 0, \tag{1.12a}$$
$$P_2 y_2'' + P_1 y_2' + P_0 y_2 = 0. \tag{1.12b}$$

We can eliminate the P_0 term between these two equations by substracting $y_2 \times$(1.12a) from $y_1 \times$(1.12b) to get

$$P_2 \left(y_1 y_2'' - y_2 y_1'' \right) + P_1 \left(y_1 y_2' - y_2 y_1' \right) = 0. \tag{1.13}$$

The term multiplying P_1 in this equation is clearly the Wronskian $W[y_1, y_2]$, and the term multiplying P_2 is the derivative of W with respect to x, i.e.

$$P_2 \frac{\mathrm{d}W}{\mathrm{d}x} + P_1 W = 0. \tag{1.14}$$

Now, *provided P_2 is nowhere zero*, we can solve for W to get

$$W(x) = \text{const} \times \exp\left(-\int \frac{P_1(x)}{P_2(x)}\,\mathrm{d}x\right). \tag{1.15}$$

Since the exponential can't vanish, it follows that *if $W = 0$ at one point, then $W \equiv 0$ everywhere* and, conversely, *if $W \neq 0$ at one point, then $W \neq 0$ everywhere.* Now we can use this result to obtain a partial converse to Proposition 1.1.

Proposition 1.2. *Two solutions of a given homogeneous second-order ODE* (H) *are linearly dependent if and only if their Wronskian is zero.*

Proof. Suppose y_1 and y_2 are two solutions of (H); if they are linearly dependent then we know already that their Wronskian is zero so now suppose for the converse that their Wronskian is zero (everywhere, by (1.15)). If y_1 is the zero function then y_1 and y_2 are certainly linearly dependent and we are done. Suppose that there is at least one value of x, say $x = a$, with $y_1(a) \neq 0$, and pick μ so that $y_2(a) = \mu y_1(a)$. Then

$$0 = W(a) = y_1(a)y_2'(a) - y_2(a)y_1'(a) = y_1(a)\left(y_2'(a) - \mu y_1'(a)\right) \tag{1.16}$$

and, since $y_1(a) \neq 0$ by assumption, we conclude that $y_2'(a) = \mu y_1'(a)$.

Now define $y(x) = y_2(x) - \mu y_1(x)$; then $y(x)$ is a solution of (H) by linearity, and satisfies the initial conditions $y(a) = 0 = y'(a)$. Thus by uniqueness of solution of (H) (Picard's Theorem: again assuming that $P_2 \neq 0$) we conclude that $y(x) \equiv 0$ and therefore y_1 and y_2 are linearly dependent. □

1.4 A basis of solutions to (H)

We can choose two particular solutions y_1 and y_2 of (H) satisfying the following initial conditions at some point $x = a$:

$$y_1(a) = 1, \quad y_1'(a) = 0, \qquad\qquad y_2(a) = 0, \quad y_2'(a) = 1. \tag{1.17}$$

By Picard's Theorem both $y_1(x)$ and $y_2(x)$ exist and are unique at least in a neighbourhood of $x = a$ provided $P_2(a) \neq 0$. Also their Wronskian has $W = 1$ at $x = a$ and so is nonzero in the same neighbourhood of $x = a$, and hence they are linearly independent.

In fact, y_1 and y_2 span the vector space of solutions. Suppose $y(x)$ is any other solution of (H) and set

$$Y(x) = y_1(x)y(a) + y_2(x)y'(a). \tag{1.18}$$

Then $Y(x)$ is also a solution of (H) and satisfies the initial conditions

$$Y(a) = y(a), \qquad\qquad Y'(a) = y'(a). \tag{1.19}$$

By uniqueness (Picard again) $Y(x) \equiv y(x)$ and thus $y(x)$ is a linear combination of y_1 and y_2. Hence they do span the vector space of solutions, i.e. they are a basis, and we conclude the following.

Proposition 1.3.

(i) The dimension of the space of solutions of H is 2.

(ii) Any pair of solutions of H with $W \neq 0$ is a basis.

Exercise: generalise everything done so far to n-th order linear ODEs.

1.5 Solution methods for homogeneous problem

There are very few general methods of solution for second-order linear ODEs of the form (H). We will discuss three well known special cases of (H) where the general solution can be found relatively easily. All three methods can be used for higher order problems with similar properties.

1.5.1 Constant coefficients

If P_2, P_1 and P_0 are constants, then (H) admits exponential solutions of the form $y(x) = \mathrm{e}^{mx}$, where m satisfies the quadratic equation $P_2 m^2 + P_1 m + P_0 = 0$, known as the *auxiliary equation*. The general solution can then easily be found as a linear combination of solutions with different values of m. Care must be taken for cases where the roots m are complex or are repeated.

1.5.2 Cauchy–Euler equation

In a Cauchy–Euler equation, the coefficients are of the form $P_2(x) = \alpha x^2$, $P_1(x) = \beta x$, $P_0(x) = \gamma$, with α, β, γ constants, so (H) takes the form

$$\alpha x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \beta x \frac{\mathrm{d}y}{\mathrm{d}x} + \gamma y = 0. \tag{1.20}$$

(Note that the "power of x" is the same in each term.) In this case, solutions can be found of the form $y(x) = x^m$, and m again satisfies a quadratic equation, $\alpha m(m-1) + \beta m + \gamma = 0$. Again, extra care is needed if the roots m are repeated or complex. An alternative approach is to make the substitution $x = \mathrm{e}^t$, which transforms (1.20) into the constant-coefficients equation

$$\alpha \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\beta - \alpha) \frac{\mathrm{d}y}{\mathrm{d}t} + \gamma y = 0. \tag{1.21}$$

1.5.3 Reduction of order

If one solution $y_1(x)$ is known, then the general solution can be found by solving an ODE of reduced order. The method is to express the solution to the ODE (H) in the form

$$y(x) = v(x) y_1(x). \tag{1.22}$$

We know that the function $v(x) = \text{const}$ is a possible answer but we seek something more general. We substitute (1.22) into (H) and simplify, using the fact that y_1 is a solution of (H), to obtain

$$P_2 y_1 v'' + (2P_2 y_1' + P_1 y_1) v' = 0, \tag{1.23}$$

which is a separable first-order ODE for v' with solution

$$v'(x) = \frac{\text{const}}{y_1(x)^2} \exp\left(-\int \frac{P_1(x)}{P_2(x)}\,\mathrm{d}x\right). \tag{1.24}$$

One further integration then gives v and thus the general solution $y(x) = v(x)y_1(x)$.

This method of constructing the general solution from a single known solution may also be derived from the expression (1.15) for the Wronskian, i.e.

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = y_1(x)^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2(x)}{y_1(x)}\right) = \text{const} \times \exp\left(-\int \frac{P_1(x)}{P_2(x)}\,\mathrm{d}x\right), \tag{1.25}$$

from which we can construct $y_2(x)$ given $y_1(x)$.

1.6 Variation of parameters

We now know a good deal about the solutions of the homogeneous ODE (H). The general solution to the inhomogeneous version (N) given by (1.6) seems to rely on us spotting a particular integral $y_{\mathrm{PI}}(x)$. The method of variation of parameters allows us to construct a solution to (N) for any forcing function f without any guesswork, provided we already know the general solution to the homogeneous equation (H).

Suppose that (H) is solved by $y(x) = c_1y_1(x) + c_2y_2(x)$ with linearly independent y_1, y_2. We seek a solution to (N) of the form

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{1.26}$$

i.e. we “vary the parameters” c_1 and c_2. First, differentiate (1.26) to find

$$y' = c_1y_1' + c_2y_2' + c_1'y_1 + c_2'y_2. \tag{1.27}$$

Now to eliminate the highest derivatives of c_i, we impose the condition

$$c_1'y_1 + c_2'y_2 = 0 \tag{1.28}$$

on c_1 and c_2. Note, since we are using two functions c_1 and c_2 to define one function y, we should have enough freedom to satisfy the additional constraint (1.28). Under the assumption (1.28), the expression (1.27) for y' simplifies to

$$y' = c_1y_1' + c_2y_2'. \tag{1.29}$$

We differentiate once more and substitute into (1.2) to get

$$\begin{aligned} \mathfrak{L}y &= P_2\left(c_1y_1'' + c_2y_2'' + c_1'y_1' + c_2'y_2'\right) + P_1\left(c_1y_1' + c_2y_2'\right) + P_0(c_1y_1 + c_2y_2) \\ &= c_1\mathfrak{L}y_1 + c_2\mathfrak{L}y_2 + P_2\left(c_1'y_1' + c_2'y_2'\right). \end{aligned} \tag{1.30}$$

But, since the y_i satisfy (H), the inhomogeneous ODE (N) becomes

$$\mathfrak{L}y = P_2\left(c_1'y_1' + c_2'y_2'\right) = f. \tag{1.31}$$

Together, (1.28) and (1.31) give two simultaneous linear equations for c_1' and c_2', namely

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f/P_2. \end{pmatrix} \tag{1.32}$$

Note that the determinant of the matrix on the left-hand side is the Wronskian W, which we know is nonzero by the assumed linear independence of y_1 and y_2. We can therefore invert (1.32) to get

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f/P_2. \end{pmatrix} = \frac{f}{P_2 W} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}. \tag{1.33}$$

We can thus integrate to obtain

$$c_1(x) = -\int^x \frac{f(\xi)y_2(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \qquad c_2(x) = \int^x \frac{f(\xi)y_1(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \tag{1.34}$$

and, by substituting into (1.26)

$$y(x) = -\int^x \frac{f(\xi)y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi + \int^x \frac{f(\xi)y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi. \tag{1.35}$$

In principle, (1.35) allows us to construct a particular solution to (N) for any right-hand side f. There is some freedom in the construction (1.35): firstly in the choice of two linearly independent solutions (y_1, y_2) of (H); and secondly in setting the lower limits in the integrals. We will show below how to use this freedom to fit boundary conditions, after doing an example.

Example 1.2. *Consider the equation*

$$y''(x) + y(x) = \tan x \quad \textit{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}. \tag{1.36}$$

The corresponding homogeneous equation is $y'' + y = 0$*, for which we may choose two linearly-independent solutions as*

$$y_1(x) = \cos x, \qquad y_2(x) = \sin x. \tag{1.37}$$

The Wronskian turns out to be

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = \cos^2 x + \sin^2 x = 1, \tag{1.38}$$

and so by (1.34) we have

$$c_1(x) = -\int \tan x \sin x\,\mathrm{d}x = \sin(x) - \log(\sec x + \tan x), \tag{1.39a}$$

$$c_2(x) = \int \tan x \cos x\,\mathrm{d}x = -\cos x. \tag{1.39b}$$

Thus a particular integral of the inhomogeneous ODE (1.36) *is given by*

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = -\cos(x)\log(\sec x + \tan x). \tag{1.40}$$

It would have been very difficult to "spot" this from (1.36)*!*

1.7 Fitting boundary conditions

We now develop a general method to solve the inhomogeneous ODE (N) with homogeneous boundary conditions. We consider the BVP

$$P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) = f(x) \quad a < x < b, \tag{1.41a}$$

with boundary data

$$y(a) = 0 = y(b). \tag{1.41b}$$

We will see later on how generalised boundary conditions more complicated than (1.41b) may be handled. We follow the Variation of Parameters recipe (1.26), but now making specific choices of the two basis solutions y_1 and y_2 such that $y_1(a) = 0$ and $y_2(b) = 0$. We assume for the moment that such y_1 and y_2 exist and are linearly independent so that $W[y_1, y_2] \neq 0$, and it follows that $y_1(b) \neq 0$ and $y_2(a) \neq 0$.

So the solution takes the form $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, with the c_i as in (1.34), and the boundary conditions (1.41b) lead to

$$y(a) = c_1(a)y_1(a) + c_2(a)y_2(a) = c_2(a)y_2(a) = 0, \tag{1.42a}$$
$$y(b) = c_1(b)y_1(b) + c_2(b)y_2(b) = c_1(b)y_1(b) = 0 \tag{1.42b}$$

with the choices made for y_i. This requires that we take $c_2(a) = 0 = c_1(b)$ and, by imposing these conditions on (1.34), we obtain explicit unique forms for c_1 and c_2, namely

$$c_1(x) = \int_x^b \frac{f(\xi)y_2(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \qquad c_2(x) = \int_a^x \frac{f(\xi)y_1(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi \tag{1.43}$$

(note the switching of the limits in the integral for c_1).

The solution to the BVP (1.41) can thus be written as

$$y(x) = \int_a^x \frac{f(\xi)y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi + \int_x^b \frac{f(\xi)y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \tag{1.44}$$

which we can write concisely as

$$y(x) = \int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi, \tag{1.45}$$

where

$$g(x,\xi) = \begin{cases} \dfrac{y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)} & a < \xi < x < b, \\[2ex] \dfrac{y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)} & a < x < \xi < b, \end{cases} \tag{1.46}$$

is called the *Green's function*. We will return to study the properties of g in more detail in Section 2.

Example 1.3. *We illustrate the construction of g for the BVP*

$$y''(x) + y(x) = f(x) \quad \textit{for} \quad 0 < x < \frac{\pi}{2}, \tag{1.47a}$$

with boundary conditions

$$y(0) = 0 = y\left(\frac{\pi}{2}\right). \tag{1.47b}$$

1. *Identify* (H) *as* $y'' + y = 0$.
2. *Choose solutions* y_1 *and* y_2 *such that* $y_1(0) = 0$ *and* $y_2(\pi/2) = 0$*:* $y_1(x) = \sin x$ *and* $y_2(x) = \cos x$ *will do.*
3. *Note* $P_2 = 1$ *and calculate* $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -1$.

Therefore (1.46) *gives the Green's function as*

$$g(x,\xi) = \begin{cases} -\sin\xi\cos x & 0 < \xi < x < \frac{\pi}{2}, \\ -\cos\xi\sin x & 0 < x < \xi < \frac{\pi}{2}. \end{cases} \tag{1.48}$$

By (1.45)*, the solution of the BVP* (1.47) *is then given by*

$$y(x) = \int_0^{\frac{\pi}{2}} g(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{1.49}$$

Example 1.4.: Nonexistence/nonuniqueness of solution

Here we consider the same ODE as in Example 1.3 *but with modified boundary conditions, namely*

$$y''(x) + y(x) = f(x) \quad \text{for} \quad 0 < x < \frac{\pi}{2}, \tag{1.50a}$$

subject to

$$y(0) = 0 = y'\left(\frac{\pi}{2}\right). \tag{1.50b}$$

The problem here is that $y_1(x) = \sin(x)$ *satisfies* both *boundary conditions* (1.50b)*, and it is impossible to find* linearly independent y_1 *and* y_2 *satisfying one boundary condition each. The construction that led to* (1.44) *therefore fails.*

However, from the discussion in §1.4*, we know that* any *solution of* (1.50a) *can be written in the form "particular integral + complementary function", that is,*

$$y(x) = \underbrace{c_1(x)y_1(x) + c_2(x)y_2(x)}_{PI} + \underbrace{\alpha y_1(x) + \beta y_2(x)}_{CF}, \tag{1.51}$$

where, as before,

$$c_1(x) = -\int_x^{\pi/2} f(\xi)y_2(\xi)\,\mathrm{d}\xi, \qquad c_2(x) = -\int_0^x f(\xi)y_1(\xi)\,\mathrm{d}\xi, \tag{1.52}$$

and α*,* β *are arbitrary constants. Here we use variation of parameters just to find the particular integral: we have not yet attempted to apply the boundary conditions. Given the condition* (1.28) *satisfied by* c_1 *and* c_2*, we can easily calculate*

$$y'(x) = \big[c_1(x)y_1'(x) + c_2(x)y_2'(x)\big] + \big[\alpha y_1'(x) + \beta y_2'(x)\big]. \tag{1.53}$$

Now we impose the boundary conditions (1.50b)*. Using the particular forms* $y_1(x) = \sin x$ *and* $y_2(x) = \cos x$ *and the conditions* $c_2(0) = 0 = c_1(\pi/2)$*, we calculate*

$$y(0) = \beta \qquad \text{and} \qquad y'(\pi/2) = -\beta - c_2(\pi/2), \tag{1.54}$$

and substitution into (1.50b) *gives* $\beta = 0$ *and* $c_2(\pi/2) = 0$*, i.e.*

$$\int_0^{\pi/2} f(\xi)\sin(\xi)\,\mathrm{d}\xi = 0. \tag{1.55}$$

The BVP (1.50) *has* no solution *unless* f *satisfies the* solvability condition (1.55)*. If* (1.55) is *satisfied, then the solution of* (1.50) *exists but is not unique, since the value of* α *in* (1.53) *remains arbitrary.*

1.8 Analogy with linear algebra

The difficulty encountered in Example 1.4 is reminiscent of a difficulty that can occur in the solution of systems of linear equations. Consider the homogeneous and inhomogeneous problems

$$A\mathbf{x} = \mathbf{0}, \tag{$\mathcal{H}$}$$

$$A\mathbf{x} = \mathbf{b}, \tag{$\mathcal{N}$}$$

where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$. If A is invertible (i.e. has nonzero determinant), then ($\mathcal{H}$) has only the trivial solution $\mathbf{x} = \mathbf{0}$. In this case, ($\mathcal{N}$) has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

However, if ($\mathcal{H}$) has a solution $\mathbf{x} = \mathbf{x}_1 \neq \mathbf{0}$, then A must be singular and, for general $\mathbf{b}$, the solution of ($\mathcal{N}$) does not exist. If for some particular choice of $\mathbf{b}$ a solution of ($\mathcal{N}$) for $\mathbf{x}$ *does* exist, then it is non-unique, since any vector of the form $\mathbf{x} + \alpha\mathbf{x}_1$ is also a solution. In summary, if the homogeneous problem admits non-trivial solutions, then the inhomogeneous problem has either no solution or an infinite number of solutions, but how can we determine which it is?

One option is to note that (since the row and column ranks of A are equal) A^* is singular if and only if A is, where A^* here denotes the transpose of A. Thinking of A as a linear transformation on $\mathbb{R}^n$, we can also identify A^* as the corresponding *adjoint* transformation, in the sense that

$$\langle A\mathbf{x}, \mathbf{w}\rangle \equiv \langle \mathbf{x}, A^*\mathbf{w}\rangle\,, \tag{1.56}$$

where $\langle \mathbf{x}, \mathbf{w}\rangle \equiv \mathbf{x}\cdot\mathbf{w}$ denotes the usual Cartesian inner product.

If ($\mathcal{H}$) admits non-trivial solutions for $\mathbf{x}$, then the corresponding *adjoint problem*

$$A^*\mathbf{w} = \mathbf{0}, \tag{$\mathcal{H}^*$}$$

also admits non-trivial solutions for $\mathbf{w}$. By taking the inner product of ($\mathcal{N}$) with $\mathbf{w}$ and using (1.56), we deduce that a necessary condition for ($\mathcal{N}$) to be solvable is that

$$\langle \mathbf{b}, \mathbf{w}\rangle = 0 \text{ for all } \mathbf{w} \text{ satisfying } (\mathcal{H}^*). \tag{1.57}$$

It can be shown that the solvability condition (1.57) is also sufficient, and hence that ($\mathcal{N}$) is solvable for $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to every vector in the kernel of A^*. Indeed, this is really just a re-phrasing of the standard result for finite-dimensional inner product spaces $\mathrm{im}(A) = \ker(A^*)^\perp$: "the image of A is the orthogonal complement of the kernel of A^*".

Collecting all the above together, we see that there are three alternative outcomes for the inhomogeneous problem ($\mathcal{N}$): there is either a unique solution, no solution, or an infinite number of solutions. These can be summarised as follows in the so-called Fredholm Alternative Theorem (FAT).

Theorem 1.1. ***Fredholm Alternative ($\mathbb{R}^n$ version)***
Exactly one of the following possibilities occurs.

1. *The homogeneous equation ($\mathcal{H}$) $A\mathbf{x} = \mathbf{0}$ has only the zero solution. In this case the solution of ($\mathcal{N}$) $A\mathbf{x} = \mathbf{b}$ is unique.*

2. *The homogeneous equation ($\mathcal{H}$) $A\mathbf{x} = \mathbf{0}$ admits non-trivial solutions, and so does ($\mathcal{H}^*$) $A^*\mathbf{w} = \mathbf{0}$. In this case there are two sub-possibilities:*

2(a) if $\langle \mathbf{b}, \mathbf{w} \rangle = 0$ *for all* $\mathbf{w}$ *satisfying* $(\mathcal{H}^*)$*, then* $(\mathcal{N})$ *has a non-unique solution;*

2(b) otherwise, $(\mathcal{N})$ *has no solution.*

Now let us see how Theorem 1.1 relates to Examples 1.3 and 1.4.

Example 1.3 corresponds to alternative 1 of Theorem 1.1. The homogeneous problem $\mathfrak{L}y = y'' + y = 0$, subject to the boundary conditions $y(0) = y(\pi/2) = 0$ has no non-trivial solutions. In this case, we are able to find two linearly independent solutions satisfying $y_1(0) = 0 = y_2(\pi/2)$, and the construction in §1.7 provides a unique solution to the inhomogeneous problem $\mathfrak{L}y = f$ for arbitrary f.

In Example 1.4, the homogeneous problem $\mathfrak{L}y = y'' + y$, subject to the new boundary conditions $y(0) = y'(\pi/2) = 0$ does admit a non-trivial solution $y_1(x) = \sin x$. In this case, it is impossible to find two linearly independent solutions satisfying $y_1(0) = 0 = y_2'(\pi/2)$, and the construction of the Green's function given in §1.1 fails. This corresponds to alternative 2 of Theorem 1.1: the inhomogeneous problem $\mathfrak{L}y = f$ either has (2a) a non-unique solution, if f satisfies the solvability condition (1.55); or (2b) no solution, if (1.55) is not satisfied. However, to understand how (1.55) relates to (1.57), we need to define the adjoint of a differential operator.

1.9 Adjoint operator and boundary conditions

We define the *inner product* between two (suitably smooth) functions defined on an interval $[a, b]$ by

$$\langle u, v \rangle := \int_a^b u(x)\overline{v(x)}\,\mathrm{d}x, \tag{1.58}$$

where the overbar denotes complex conjugate. Where it is clear that we are dealing with real-valued functions, we will generally drop the overbar for simplicity.

In general, for a given linear operator $\mathfrak{L}$, the corresponding *adjoint operator* $\mathfrak{L}^*$ is defined by the inner product relation

$$\langle \mathfrak{L}y, w \rangle = \langle y, \mathfrak{L}^* w \rangle \tag{1.59}$$

for all y, w in a suitable inner product space. To determine the adjoint of a linear differential operator, one needs (i) to move the derivatives of the operator from y to w, using integration by parts, and (ii) to set the boundary conditions to ensure that all boundary terms vanish.

Example 1.5. *Let*

$$\mathfrak{L}y = y'' \tag{1.60}$$

for $a \le x \le b$*. We use integration by parts to calculate*

$$\begin{aligned}\langle \mathfrak{L}y, w \rangle = \int_a^b y''(x)w(x)\,\mathrm{d}x &= -\int_a^b y'(x)w'(x)\,\mathrm{d}x + \left[y'(x)w(x)\right]_a^b \\ &= \int_a^b y(x)w''(x)\,\mathrm{d}x + \left[y'(x)w(x) - y(x)w'(x)\right]_a^b \equiv \langle y, \mathfrak{L}^* w \rangle\,. \end{aligned} \tag{1.61}$$

To enforce this identity, we identify the integrand in (1.61) *with* $\mathfrak{L}^* w$*, i.e.*

$$\mathfrak{L}^* w = w''. \tag{1.62}$$

We note in this case that $\mathfrak{L} \equiv \mathfrak{L}^*$*: the operator is* self-adjoint.

We must also ensure that the boundary terms in (1.61) *vanish. Thus, the boundary conditions imposed on* y *imply corresponding* adjoint boundary conditions *to be imposed on* w.

As a first illustration, suppose that y *satisfies the boundary conditions*

$$\mathfrak{B}_1 y = y(a) = 0, \qquad \mathfrak{B}_2 y = y(b) = 0. \tag{BC1}$$

Then the boundary terms in (1.61) *reduce to*

$$y'(b)w(b) - y'(a)w(a) - y(b)w'(b) + y(a)w'(a) = y'(b)w(b) - y'(a)w(a) \tag{1.63}$$

and, to ensure that this vanishes for all $y'(a)$ *and* $y'(b)$, *we deduce the adjoint boundary conditions*

$$\mathfrak{B}_1^* w = w(a) = 0, \qquad \mathfrak{B}_2^* w = w(b) = 0. \tag{BC1*}$$

Alternatively, if we impose the more complicated boundary condtions

$$\mathfrak{B}_1 y = y'(a) = 0, \qquad \mathfrak{B}_2 y = 3y(a) - y(b) = . \tag{BC2}$$

on y, *then the boundary terms in* (1.61) *may be expressed in the form*

$$y'(b)w(b) - y'(a)w(a) - y(b)w'(b) + y(a)w'(a) = y(a)w'(a) - 3y(a)w'(b) + y'(b)w(b). \tag{1.64}$$

To ensure that this expression vanishes for all $y(a)$ *and* $y'(b)$, *we deduce the adjoint boundary conditions*

$$\mathfrak{B}_1^* w = w'(a) - 3w'(b) = 0, \qquad \mathfrak{B}_2^* w = w(b) = 0. \tag{BC2*}$$

Example 1.5 illustrates the following general points about the adjoint of a linear differential operator.

(i) We can calculate the adjoint $\mathfrak{L}^*$ of an operator $\mathfrak{L}$ without worrying about the boundary conditions.

(ii) If $\mathfrak{L}^* = \mathfrak{L}$, then the operator $\mathfrak{L}$ is *self-adjoint*.

(iii) When $\mathfrak{L}$ is supplemented with homogeneous boundary conditions to give a problem of the form $(\mathfrak{L} + \text{BC})$, then corresponding *adjoint boundary conditions* are generated to gave an *adjoint problem* $(\mathfrak{L}^* + \text{BC}^*)$.

(iv) If $\mathfrak{L} = \mathfrak{L}^*$ and $\text{BC} = \text{BC}^*$ then the problem is said to be *fully self-adjoint* (as in the case (BC1) above).

(v) As illustrated by (BC2) and (BC2*), it is possible for the *operator* to be self-adjoint but the boundary conditions not to be (sometimes this case is called "formally self-adjoint").

By following through the integration by parts procedure, one can find a general form for the adjoint operator:

$$\mathfrak{L}y = P_2 y'' + P_1 y' + P_0 y \tag{1.65a}$$

$$\Leftrightarrow \mathfrak{L}^* w = (P_2 w)'' - (P_1 w)' + P_0 w. \tag{1.65b}$$

One can easily check that an analagous procedure works for higher-order operators: to find the adjoint, move all the coefficients inside the derivatives, and switch the sign of any odd-ordered derivatives. Using (1.65), we calculate

$$\begin{aligned} w\mathfrak{L}y - y\mathfrak{L}^* w &= w\left[P_2 y'' + P_1 y' + P_0 y\right] - y\left[(P_2 w)'' - (P_1 w)' + P_0 w)\right] \\ &= \left[P_2 w y' - (P_2 w)' y + P_1 w y\right]' \end{aligned} \tag{1.66}$$

and therefore

$$\langle \mathfrak{L}y, w\rangle - \langle y, \mathfrak{L}^* w\rangle = \big[P_2 w y' - (P_2 w)' y + P_1 w y\big]_a^b. \tag{1.67}$$

Given appropriate homogeneous boundary conditions for y, we can deduce the corresponding adjoint boundary conditions for w by setting the final integrated term in (1.67) equal to zero. This integrated term must then be expressible in the form

$$\begin{aligned}\langle \mathfrak{L}y, w\rangle - \langle y, \mathfrak{L}^* w\rangle &= \big[P_2 w y' - (P_2 w)' y + P_1 w y\big]_a^b \\ &= (K_1^* w)(\mathfrak{B}_1 y) + (K_2^* w)(\mathfrak{B}_2 y) + (K_1 y)(\mathfrak{B}_1^* w) + (K_2 y)(\mathfrak{B}_2^* w),\end{aligned} \tag{1.68}$$

where $K_1 y$ and $K_2 y$ are linearly independent of $\mathfrak{B}_1 y$ and $\mathfrak{B}_2 y$, and likewise $K_1^* w$ and $K_2^* w$ are linearly independent of $\mathfrak{B}_1^* w$ and $\mathfrak{B}_2^* w$. For example, in the case of (BC2) from Example 1.5, we can write

$$\big[y'w - yw'\big]_a^b = \underbrace{-w(a)}_{K_1^* w}\underbrace{y'(a)}_{\mathfrak{B}_1 y} + \underbrace{w'(b)}_{K_2^* w}\underbrace{\big(3y(a) - y(b)\big)}_{\mathfrak{B}_2 y} + \underbrace{y(a)}_{K_1 y}\underbrace{\big(w'(a) - 3w'(b)\big)}_{\mathfrak{B}_1^* w} + \underbrace{y'(b)}_{K_2 y}\underbrace{w(b)}_{\mathfrak{B}_2^* w}. \tag{1.69}$$

We then see how the given boundary conditions $\mathfrak{B}_1 y = \mathfrak{B}_2 y = 0$ enforce the corresponding ajoint conditions $\mathfrak{B}_1^* w = \mathfrak{B}_2^* w = 0$.

Expanding out $\mathfrak{L}^*$ in (1.65), we find

$$\mathfrak{L}^* w = P_2 w'' + \big(2P_2' - P_1\big)\, w' + \big(P_2'' - P_1' + P_0\big)\, w, \tag{1.70}$$

and, by comparing with $\mathfrak{L}$, we deduce that $\mathfrak{L}$ is self-adjoint if and only if $P_1 = P_2'$. If so then, setting $P_2 = -p$, $P_1 = -p'$ and $P_0 = q$, we can write $\mathfrak{L}$ as

$$\mathfrak{L}y = -\big(py'\big)' + qy, \tag{1.71}$$

which is the most general formally self-adjoint second-order differential operator.

Finally, we are ready for a statement (without proof!) of the Fredholm Alternative Theorem (FAT) for linear differential operators of the form (1.65a).

Theorem 1.2. ***Fredholm Alternative (linear ODE version)***

We consider the linear homogeneous and inhomogeneous ODEs

$$\mathfrak{L}y = 0, \tag{H}$$

$$\mathfrak{L}y = f \not\equiv 0, \tag{N}$$

for $0 < x < a$, supplemented by linear homogeneous boundary conditions of the form

$$\left.\begin{aligned}\mathfrak{B}_1 y = \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = 0, \\ \mathfrak{B}_2 y = \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = 0,\end{aligned}\right\} \tag{BC}$$

(with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $(\alpha_3, \alpha_4, \beta_3, \beta_4)$ linearly independent). We also define the homogeneous adjoint equation

$$\mathfrak{L}^* w = 0, \tag{H*}$$

and corresponding adjoint boundary conditions (BC*)*, computed as described above.*

Exactly one of the following possibilities occurs.

1. *The homogeneous problem* (H+BC) *has only the zero solution. In this case the solution of* (N+BC) *is unique.*

2. *The homogeneous problem* (H+BC) *admits non-trivial solutions, and so does* (H*+BC*). *In this case there are two sub-possibilities:*

   2(a) if $\langle f, w\rangle = 0$ *for all* w *satisfying* (H*+BC*), *then* (N+BC) *has a non-unique solution;*

   2(b) otherwise, (N+BC) *has no solution.*

Exercise: Demonstrate that Examples 1.3 and 1.4 are consistent with FAT.

1.10 Inhomogeneous boundary conditions and FAT

Our statement of the Fredholm Alternative in Theorem 1.2 concerns ODEs subject to homogeneous boundary conditions. A little more work is required to apply the results to problems with inhomogeneous boundary conditions. Suppose that we replace the boundary conditions (BC) with

$$\left.\begin{aligned}\mathfrak{B}_1 y = \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = \gamma_1,\\ \mathfrak{B}_2 y = \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = \gamma_2,\end{aligned}\right\} \tag{NBC}$$

for some constants γ_1 and γ_2. First we note that the condition for a unique solution of the modified problem (N+NBC) is exactly the same as case 1 in Theorem 1.2. To see this, let $v(x)$ be any twice differentiable function that satisfies the conditions (NBC): it need not be a solution of the ODE (H). We can then make the boundary conditions homogeneous by subtracting off $v(x)$, i.e. defining $\tilde{y}(x) = y(x) - v(x)$, so that $\tilde{y}$ satisfies the problem

$$\mathfrak{L}\tilde{y} = f - \mathfrak{L}v = \tilde{f}, \tag{1.72}$$

say, with homogeneous boundary conditions $\mathfrak{B}_1\tilde{y} = 0 = \mathfrak{B}_2\tilde{y}$. We can now apply FAT to deduce that there is a unique solution for $\tilde{y}$, and therefore also for y, if and only if the homogeneous problem (H+BC) has no non-trivial solutions.

If (H+BC) *does* admit non-trivial solutions, then we can apply Case 2 of FAT to deduce that there is no solution unless $\langle \tilde{f}, w\rangle = 0$ for all w in the kernel of (H*+BC*), in which case the solution is non-unique. The solvability condition in this case may be expressed as

$$\begin{aligned}0 &= \langle \tilde{f}, w\rangle = \langle f, w\rangle - \langle \mathfrak{L}v, w\rangle\\ &= \langle f, w\rangle - \langle v, \mathfrak{L}^*w\rangle - (K_1^*w)(\mathfrak{B}_1 v) - (K_2^*w)(\mathfrak{B}_2 v) - (K_1 v)(\mathfrak{B}_1^* w) - (K_2 v)(\mathfrak{B}_2^* w),\end{aligned} \tag{1.73}$$

when we apply the decomposition (1.68). Since w satisfies the homogeneous adjoint problem (H*+BC*), the right-hand side of (1.73) only involves functions of v that are known by the given boundary conditions $\mathfrak{B}_1 v = \gamma_1$ and $\mathfrak{B}_2 v = \gamma_2$, and we thus deduce the solvability condition

$$\langle f, w\rangle = \gamma_1 K_1^* w + \gamma_2 K_2^* w. \tag{1.74}$$

We note that (1.74) does not involve the function v that was introduced to make the boundary conditions homogeneous, and indeed one can obtain (1.74) directly without first simplifying the boundary conditions. As above, let w be any solution of the homogeneous adjoint problem (H*+BC*), and take the inner product of (N) with w to get

$$\langle f, w\rangle = (K_1^* w)(\mathfrak{B}_1 y) + (K_2^* w)(\mathfrak{B}_2 y) + (K_1 y)(\mathfrak{B}_1^* w) + (K_2 y)(\mathfrak{B}_2^* w). \tag{1.75}$$

Application of the relevant boundary conditions then immediately produces (1.74).

In summary, when the boundary conditions are inhomogeneous, we have shown the following.

- The condition for a unique solution to exist (Case 1 of FAT) is unaffected.
- For cases where there is not a unique solution, the solvability condition is still obtained by taking the inner product with a non-trivial solution w of the homogeneous adjoint problem. Now the boundary terms produced by integration by parts do not disappear identically but do only involve quantites that are in principle known from the specified boundary conditions.

Example 1.6. Solve $y''(x) = f(x)$ on $0 < x < 1$ with $y(0) = 0$ and $y'(1) = 7$.

Here $\mathfrak{L}$ is self-adjoint, and the homogeneous adjoint problem is $L^* w = w'' = 0$ with $w(0) = w'(1) = 0$. This only has the trivial solution $w \equiv 0$, so original BVP has a unique solution for any $f(x)$.

For this simple ODE, we can construct the solution straightforwardly as follows. First let's make the boundary conditions homogeneous by subtracting off a suitable solution of the homogeneous problem, namely $u(x) = 7x$. Thus $\tilde{y} = y - u$ satisfies

$$\tilde{y}''(x) = f(x) \quad \text{on } 0 < x < 1, \qquad \tilde{y}(0) = 0 = \tilde{y}'(1). \tag{1.76}$$

We can easily integrate this simple ODE directly; alternatively, the Green's function for this problem is easily found to be given by

$$g(x,\xi) = \begin{cases} -x & 0 < x < \xi < 1, \\ -\xi & 0 < \xi < x < 1, \end{cases} \tag{1.77}$$

and the solution of the BVP is then

$$y(x) = 7x + \int_0^1 g(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{1.78}$$

Example 1.7. Solve the same ODE $y'' = 3$ with boundary conditions $y'(0) = 0$ and $y'(1) = \beta$.

The problem is again self-adjoint. The homogeneous adjoint problem $w'' = 0$, $w'(0) = 0 = w'(1)$ has the non-trivial solution $w = 1$ (or any multiple thereof). Now calculate

$$\begin{aligned} \langle y'', w\rangle &= \langle f, w\rangle \\ \Rightarrow \quad \int_0^1 y''(x)\,\mathrm{d}x &= \int_0^1 3\,\mathrm{d}x = 3 \\ \Rightarrow \quad \big[y'\big]_0^1 &= \beta = 3 \end{aligned} \tag{1.79}$$

Thus if $\beta \neq 3$, we have a contradiction and no solution exists, while if $\beta = 3$, we have a non-unique solution.

Example 1.8. When is the BVP

$$y''(x) + y(x) = f(x) \text{ for } 0 < x < \frac{\pi}{2}, \qquad y(0) = 1, \qquad y'\left(\frac{\pi}{2}\right) = 0 \tag{1.80}$$

solvable for y?

This is a very slightly altered version of Example 1.4. *The problem is again self-adjoint, and we know that* $w(x) = \sin x$ *satisfies the homogeneous problem. So take the inner product with* $\sin x$ *and integrate by parts to get*

$$\int_0^{\pi/2} \big(y''(x) + y(x)\big) \sin x \, \mathrm{d}x \equiv [y'(x) \sin x - y(x) \cos x]_0^{\pi/2} = 1, \tag{1.81}$$

when we evaluate the right-hand side using the given boundary conditions. The solvability condition in this case is therefore

$$\int_0^{\pi/2} f(x) \sin x \, \mathrm{d}x = 1. \tag{1.82}$$