5 Special functions

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5.1 Introduction

We have seen in the previous section a method to construct power series solutions to ODEs with non-constant coefficients and singular points. Except for a few examples, even if a closed form for the coefficients a_k can be found, the resulting power series cannot be expressed in terms of elementary functions, i.e. exponentials, sines, cosines, etc. Nevertheless, some particular ODEs occur frequently enough for their solutions to have been given special names and for their behaviour to be fully characterised. In this section, we explore some of these so-called *special functions*.

5.2 Bessel functions

5.2.1 Bessel's equation

Bessel's equation arises whenever one separates the variables in the Laplacian in cylindrical polar coordinates. For example, consider the vibrating membrane of a circular drum. It may be shown that the transverse displacement $w(x, y, t)$ of the membrane at time t and position (x, y) satisfies the two-dimensional *wave equation*

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \tag{5.1}$$

where c is a constant (representing the wave-speed and given by $c = \sqrt{T/\rho}$, where T and ρ are the membrane tension and density). If the membrane is pinned at the boundary of a disk of radius a, then we have to solve (5.1) in $x^2 + y^2 < a^2$, subject to the boundary condition

$$w = 0 \qquad \text{at } x^2 + y^2 = a^2. \tag{5.2}$$

We look for a *normal mode* in which the membrane oscillates with frequency ω, so that the displacement takes the form $w(x, y, t) = u(x, y)\cos(\omega t + \phi)$. By substituting into (5.1), we find that u satisfies the *Helmholtz equation*

$$\nabla^2 u + \lambda u = 0, \tag{5.3}$$

with $\lambda = \omega^2/c^2$.

Now let us switch to plane polar coordinates (r, θ) such that $(x, y) = r(\cos\theta, \sin\theta)$, and thus obtain the equation and boundary condition:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0 \qquad 0 \leq r < a, \quad 0 \leq \theta \leq 2\pi, \tag{5.4a}$$

$$u = 0 \qquad r = a, \quad 0 \leq \theta \leq 2\pi, \tag{5.4b}$$

$$u \ 2\pi\text{-periodic in } \theta. \tag{5.4c}$$

This is a PDE eigenvalue problem: $u \equiv 0$ always satisfies the problem (5.4), and our aim is to find values of λ such that there are non-trivial solutions $u(r, \theta)$.

Since $u(r, \theta)$ is periodic in θ we can expand u into a Fourier series of the form

$$u(r, \theta) = U_0(r) + \sum_{n=1}^{\infty} U_n(r)\cos n\theta + V_n(r)\sin n\theta. \tag{5.5}$$

Substitution of (5.5) into (5.4) gives

$$\frac{1}{r}\left(rU_n'(r)\right)' + \left(\lambda - \frac{n^2}{r^2}\right)U_n(r) = 0, \qquad \text{for } 0 \leq r < a, \tag{5.6a}$$

$$U_n(r) = 0 \qquad \text{at } r = a. \tag{5.6b}$$

The same equation and boundary condition hold for $V_n(r)$. Now eliminate λ by the rescaling $U_n(r) = y(x)$ with $x = \lambda^{1/2} r$, resulting in

$$x^2 y''(x) + xy'(x) + \left(x^2 - n^2\right)y(x) = 0, \tag{5.7}$$

which is known as *Bessel's equation* of order n.

5.2.2 Bessel functions of first and second kind

Bessel's equation (5.7) has a regular singular point at $x = 0$, with indicial equation given by $F(\alpha) = \alpha^2 - n^2 = 0$, the solutions of which are $\alpha_1 = n$, $\alpha_2 = -n$, with a double root for $n = 0$. In general, the parameter n in (5.7) can be any complex number but, in the context described above where $u(r, \theta)$ is required to be 2π-periodic in θ, we need only consider n to be a non-negative integer. Similarly, since x is a scaled version of the radial coordinate r, we focus on non-negative values of x. A detailed analysis of the singular point at $x = 0$, as in §4.2, reveals that one solution of (5.7) is given by a Frobenius series about $x = 0$ with the exponent $\alpha_1 = n$, and the other solution is given by a Frobenius series with exponent $\alpha_2 = -n$ plus $\log(x)$ times the first solution (i.e. Case III(a) from §4.2.5).

The first Frobenius series, with a specific normalisation of the leading coefficient in the expansion, defines the *Bessel functions of first kind*

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k}, \tag{5.8}$$

for integer $n \geq 0$.

Similarly, a specifically normalised choice for the second series solution defines the *Bessel functions of second kind*

$$Y_n(x) = \frac{2}{\pi}\log\left(\frac{x}{2}\right)J_n(x) - \frac{1}{\pi}\left(\frac{2}{x}\right)^n \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x^2}{4}\right)^k \\ - \frac{1}{\pi}\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty}\frac{\left[\psi(k+1)+\psi(n+k+1)\right]}{k!(n+k)!}\left(-\frac{x^2}{4}\right)^k, \quad (5.9)$$

where the *digamma function* $\psi(m)$ for integer $m \geq 1$ is given by $\psi(m) = -\gamma + \sum_{k=1}^{m-1} k^{-1}$, and $\gamma = 0.5772\cdots$ is the Euler–Mascheroni constant. More details regarding the expansions (5.8) and (5.9) are explored on Problem Sheet 3.

5.2.3 Properties of Bessel functions

The first few Bessel functions $J_n(x)$ and $Y_n(x)$ are plotted in Figure 5.1. Many properties of the Bessel functions are known — see for example the NIST Digital Library of Mathematical Functions (DLMF). — and we list here just a few.

(i) Since Bessel's equation (5.7) has only one singular point for finite x, the series (5.8) and (5.9) for J_n and in Y_n have infinite radius of convergence.

(ii) Also, J_n and Y_n are oscillating functions that decay slowly as $x \to \infty$. Each has an infinite set of discrete zeros in $x > 0$, which are quite useful and have therefore been tabulated, for example at mathworld. The first few zeros of J_n and Y_n (denoted $j_{n,1}$, $j_{n,2}$, ... and $y_{n,1}$, $y_{n,2}$, ...) are listed in Table 5.1, and 5.2, respectively.

(iii) As $x \to 0$, the behaviours of the two kinds of Bessel function are quite different. For the first kind, we have $J_n(0) = 0$ if $n > 0$, and $J_0(0) = 1$, while the second kind Bessel functions are singular, with $Y_n(x) \to -\infty$ as $x \to 0$. (The singularity is logarithmic when $n = 0$, or has $Y_n(x) = O\left(x^{-n}\right)$ when $n > 0$.)

(iv) The following two recursion relations can be derived from the local expansion (5.8):

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x), \qquad J_{n+1}(x) = -2J_n'(x) + J_{n-1}(x). \quad (5.10)$$

The same relations also hold for the second-kind Bessel functions Y_n.

5.2.4 Normal modes of a circular drum

We can now express the general solution to (5.6a) in terms of Bessel functions as

$$U_n(r) = C_1 J_n\left(\lambda^{1/2} r\right) + C_2 Y_n\left(\lambda^{1/2} r\right), \quad (5.11)$$

for some arbitrary constants C_1 and C_2. We require the displacement to be bounded as $r \to 0$, and must therefore set $C_2 = 0$ to remove the singularity in Y_n. For a non-trivial solution we must have $C_1 \neq 0$, and the boundary condition (5.6b) at $r = a$ therefore leads to

$$J_n\left(\lambda^{1/2} a\right) = 0, \quad (5.12)$$

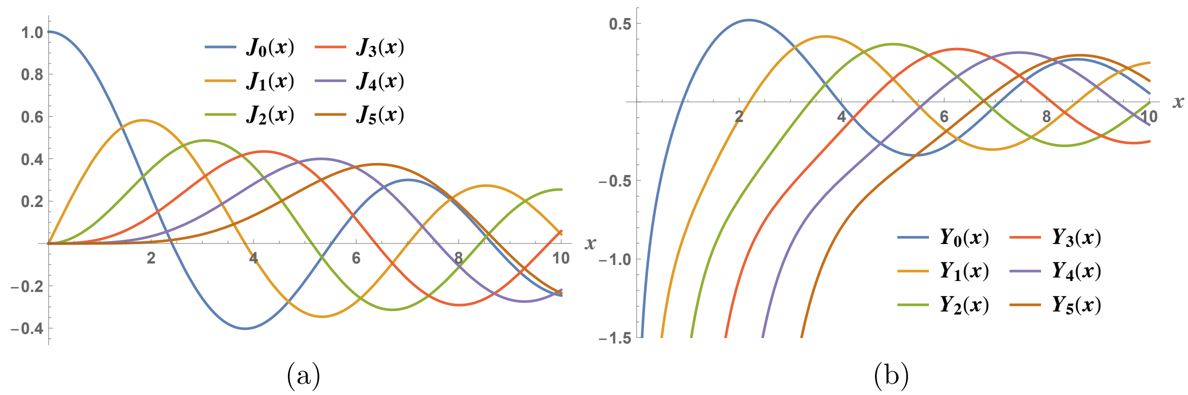

Figure 5.1: (a) Bessel functions of the first kind $J_n(x)$. (b) Bessel functions of the second kind $Y_n(x)$.

m	$j_{0,m}$	$j_{1,m}$	$j_{2,m}$	$j_{3,m}$	$j_{4,m}$
1	2.40483	3.83171	5.13562	6.38016	7.58834
2	5.52008	7.01559	8.41724	9.76102	11.0647
3	8.65373	10.1735	11.6198	13.0152	14.3725
4	11.7915	13.3237	14.796	16.2235	17.616
5	14.9309	16.4706	17.9598	19.4094	20.8269

Table 5.1: The first five zeros of J_n with $n = 0, 1, 2, 3, 4$.

m	$y_{0,m}$	$y_{1,m}$	$y_{2,m}$	$y_{3,m}$	$y_{4,m}$
1	0.893577	2.19714	3.38424	4.52702	5.64515
2	3.95768	5.42968	6.79381	8.09755	9.36162
3	7.08605	8.59601	10.0235	11.3965	12.7301
4	10.2223	11.7492	13.21	14.6231	15.9996
5	13.3611	14.8974	16.379	17.8185	19.2244

Table 5.2: The first five zeros of Y_n with $n = 0, 1, 2, 3, 4$.

i.e. $\lambda^{1/2}a$ has to be one of the zeros $j_{n,m}$ of J_n. Thus the eigenvalues for (5.6) are given by

$$\lambda = \frac{j_{n,m}^2}{a^2}, \qquad n = 0, 1, \ldots, \quad m = 1, 2, \ldots \tag{5.13}$$

with corresponding eigenfunctions

$$U_{n,m}(r) = J_n\left(j_{n,m} r/a\right). \tag{5.14}$$

We can then read off the *normal frequencies* of the drum from the definition of λ, i.e.

$$\omega_{n,m} = j_{n,m}\,\frac{c}{a}. \tag{5.15}$$

5.2.5 Sturm–Liouville form

The differential equation (5.6a) can be written in Sturm-Liouville form by multiplying through by r. For convenience we also pose the problem on the unit interval (corresponding to the modes in a disk of unit radius $a = 1$, which may be obtained by rescaling r with a) to get the eigenvalue problem

$$\mathfrak{L}U(r) = -\left(rU'(r)\right)' + \frac{n^2}{r}\,U(r) = \lambda r U(r), \qquad \text{for } 0 < r < 1, \tag{5.16a}$$

$$U(r) = 0 \qquad \text{at } r = 1, \tag{5.16b}$$

$$U(r) \text{ bounded} \qquad \text{as } r \to 0. \tag{5.16c}$$

From above, we know that the eigenvalues and eigenfunctions for (5.16) are given by

$$\lambda_{n,m} = j_{n,m}^2, \qquad U_{n,m}(r) = J_n\left(j_{n,m} r\right). \tag{5.17}$$

We recognise (5.16a) as a singular Sturm–Liouville equation with weighting function r, and thus deduce the following orthogonality relation between eigenfunctions:

$$\int_0^1 J_n\left(j_{n,\ell} r\right) J_n\left(j_{n,m} r\right) r \,\mathrm{d}r = 0 \qquad \text{for } \ell \neq m. \tag{5.18}$$

A separate calculation for the case $\ell = m$ results in [*see Problem Sheet 3*]

$$\int_0^1 J_n^2\left(j_{n,m} r\right) r \,\mathrm{d}r = \frac{1}{2}\left(J_n'(j_{n,m})\right)^2. \tag{5.19}$$

5.3 Legendre functions

5.3.1 The Legendre equation

The Legendre equation arises when studying eigenvalue problems for the 3D Laplacian operator in spherical coordinates. Suppose again we are solving the Helmholtz equation (5.3) but now using spherical polars (r, θ, ϕ), so the Laplacian is given by

$$\nabla^2 u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = -k^2 u. \tag{5.20}$$

When we separate the variables by seeking a solution of the form

$$u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi), \tag{5.21}$$

then (5.20) may be rearranged to

$$\frac{r\big(rR(r)\big)''}{R(r)} + k^2r^2 = -\frac{(\sin\theta\,\Theta'(\theta))'}{\sin\theta\,\Theta(\theta)} - \frac{\Phi''(\phi)}{\sin^2\theta\,\Phi(\phi)}. \tag{5.22}$$

By the usual argument, the left-hand side of (5.22) is a function only of r, while the right-hand side is independent of r, so they must both equal a constant, λ say. We then have

$$-\frac{\Phi''(\phi)}{\Phi(\phi)} = \frac{\sin\theta\,(\sin\theta\,\Theta'(\theta))'}{\Theta(\theta)} + \lambda\sin^2\theta, \tag{5.23}$$

which likewise must equal a constant. For Φ to be a 2π-periodic function, that constant must be of the form m^2, where $m \geq 0$ is an integer: this gives $\Phi = \text{const}$ if $m = 0$ or $\Phi(\phi) = \cos(m\phi + \alpha)$ if $m > 0$. We are then left with the following linear ODE for $\Theta(\theta)$:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{5.24}$$

Equation (5.24) is to be solved for $0 < \theta < \pi$. It may readily be verified that $\theta = 0$ and $\theta = \pi$ are both regular singular points of (5.24), and to get physically resonable solutions we must insist that $\Theta(\theta)$ is sufficiently well-behaved as $\theta \to 0,\ \pi$.

We can express (5.24) in a more helpful form by making the change of variable $\cos\theta = x$ and $\Theta(\theta) = y(x)$. Then $\mathrm{d}/\mathrm{d}\theta = -\sin\theta\,\mathrm{d}/\mathrm{d}x$, and (5.24) is transformed into the *associated Legendre equation* for $y(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0. \tag{5.25}$$

The parameters m and λ in (5.25) can in general take any complex values. We will focus on the case where m is a non-negative integer and (for reasons that will become clear below) $\lambda = \ell(\ell+1)$, where ℓ is also a non-negative integer. The solutions of the associated Legendre equation (5.25) are the *associated Legendre functions*; for $m = 0$, we drop the "associated" and speak of the Legendre equation and Legendre functions.

5.3.2 Properties of Legendre functions

Many properties and relations satisfied by solutions of (5.25) may be found, for example, at DLMF or mathworld. Here we list a few useful properties.

(i) The points $x = \pm 1$ and $x = \infty$ are regular singular points of the associated Legendre equation (5.25). The indicial exponents for $x = \pm 1$ are $-m/2$ and $m/2$. Thus, the local expansion yields one bounded and one unbounded solution as $x \to 1$, and similarly as $x \to -1$. (When $m = 0$, there is a repeated root of the indicial equation, and one solution is of order $\log(x \mp 1)$ as $x \to \pm 1$.)

(ii) If we consider bounded solutions of (5.25) on $-1 < x < 1$, we see that boundedness imposes *two* conditions, one at either end of the interval. This suggests that (5.25) can be posed as a singular Sturm–Liouville problem:

$$-\left(\left(1-x^2\right)y'(x)\right)' + \frac{m^2}{1-x^2}\,y(x) = \lambda y(x) \qquad \text{for } -1 < x < 1, \tag{5.26a}$$

$$y(x) \text{ bounded} \qquad \text{as } x \to \pm 1. \tag{5.26b}$$

(iii) The eigenvalues of (5.26) are given by $\lambda = \ell(\ell+1)$ with integer $\ell \geq m \geq 0$. The eigenfunctions are the corresponding associated Legendre functions, which are denoted by $y(x) = P_\ell^m(x)$. From Sturm–Liouville theory, we infer the orthogonality relation

$$\int_{-1}^{1} P_k^m(x) P_\ell^m(x)\,\mathrm{d}x = 0 \qquad \text{for } k \neq \ell. \tag{5.27}$$

The case $k = \ell$ requires explicit calculation: see Problem Sheet 3.

(iv) For $m = 0$ and integer $\ell \geq 0$, the Legendre functions (without "associated") are denoted by $P_\ell(x)$. It may be shown that P_ℓ is a polynomial of degree ℓ: if one seeks the solution of (5.25) as a power series expansion about $x = 0$,

$$y(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{5.28}$$

then the series *terminates*, with $a_k \equiv 0$ for $k > \ell$. The resulting *Legendre polynomials* are given explictly by the *Rodrigues' formula*:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left[(x^2-1)^\ell\right]. \tag{5.29}$$

(v) A second, linearly independent, solution of the Legendre equation (5.26a) with $m = 0$ is given by the Legendre function of second kind, denoted by $Q_\ell(x)$. These solutions are unbounded as $x \to \pm 1$. For the case $\ell = 0$, the solution Q_0 is found on Problem Sheet 2:

$$Q_0(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right). \tag{5.30}$$

(vi) For the general case of nonzero $m \leq \ell$, the associated Legendre functions of first and second kind are given by

$$P_\ell^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{\mathrm{d}^m P_\ell(x)}{\mathrm{d}x^m}, \tag{5.31a}$$

$$Q_\ell^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{\mathrm{d}^m Q_\ell(x)}{\mathrm{d}x^m}. \tag{5.31b}$$

The associated Legendre function P_ℓ^m is a polynomial if and only if m is even.

5.4 Generalisation: orthogonal polynomials

There are many other second order linear ODEs with families of orthogonal polynomials as solutions, satisfying orthogonality relations

$$\int_a^b p_m(x)p_n(x)r(x)\,\mathrm{d}x = 0 \qquad m \neq n \tag{5.32}$$

with a fixed weighting function $r(x)$ which can by inferred by formulating an appropriate Sturm–Liouville eigenvalue problem. One can in fact give a complete classification of all infinite families of orthogonal polynomials that can arise from second-order linear ODEs. The most important ones include the following.

1. The "Jacobi-like" polynomials, which include the Legendre, Chebyshev, and Gegenbauer polynomials, arise from ODEs of the type

$$\left(1-x^2\right)y''(x) + (a+bx)y'(x) + \lambda y(x) = 0, \tag{5.33}$$

   posed on the interval $[-1,1]$, with constants a and b and an appropriate discrete set of values of λ.

2. The associated Laguerre polynomials satisfy Laguerre's equation:

$$xy''(x) + (a+1-x)y'(x) + \lambda y(x) = 0, \tag{5.34}$$

   which admits a polynomial solution $y(x) = L_n^a(x)$ when λ is a non-negative integer n. They satisfy the orthogonality relation

$$\int_0^\infty L_m^a(x)L_n^a(x)x^a\mathrm{e}^{-x}\,\mathrm{d}x = 0 \quad \text{for } m \neq n. \tag{5.35}$$

   The Laguerre polynomials (without "associated") correspond to $a = 0$ and are denoted by $L_n(x) \equiv L_n^0(x)$.

3. Hermite polynomials are solutions of the Hermite equation

$$y''(x) - 2xy'(x) + \lambda y(x) = 0, \tag{5.36}$$

   which admits a polynomial solution $H_n(x)$ when $\lambda = 2n$ for integer $n \geq 0$. Hermite polynomials satisfy the orthogonality relation

$$\int_{-\infty}^\infty H_m(x)H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = 0 \quad \text{for } m \neq n. \tag{5.37}$$