

Problem Sheet 3

1. Frobenius method. Consider the differential equation

$$(x - 1)y''(x) - xy'(x) + y(x) = 0$$

- (a) Determine the appropriate form of the series expansion about $x = 1$ for two linearly independent solutions. [You do not need to compute the coefficients.]
 (b) Use (a) to obtain one closed form solution (i.e. not in the form of an infinite series). [Hint: Consider the choice of coefficients in the second Frobenius series.]

2. The point $x = \infty$. Consider the differential equation

$$x^3y''(x) + y(x) = 0. \quad (\star)$$

- (a) Use the transformation of variables $x = 1/t$ to show that (\star) has a regular singular point at $x = \infty$ and determine the indicial exponents.
 (b) Obtain the first Frobenius solution in the form of an infinite series in powers of t , i.e. solve explicitly for the coefficients.
 (c) Find the form of the second Frobenius solution and obtain (but do not attempt to solve) a recurrence relation for the coefficients in the series.

3. Bessel functions. Consider *Bessel's differential equation* (of order n):

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0, \quad (\dagger)$$

for integer $n > 0$.

- (a) Find the indicial exponents α_1, α_2 (with $\text{Re } \alpha_1 > \text{Re } \alpha_2$) for the local series expansion of (\dagger) about $x = 0$.
 (b) Determine the series $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha_1}$ that solves (\dagger) , giving the coefficients a_k in closed form. Find a_0 such that the series is the expansion of the *Bessel functions of first kind*,

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!(k+n)!}. \quad (\#)$$

- (c) Using $(\#)$, show that the following recursion relation is true for all integers $n \geq 0$:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

- (d) For any integer $n \geq 0$, show that

$$\int_0^1 x [J_n(\alpha x)]^2 dx = \frac{1}{2} [J_n'(\alpha)]^2,$$

where α is a zero of J_n . [Hint: Substitute $z = \alpha x$, integrate by parts, and use the fact that J_n satisfies Bessel's equation.]

4. Bessel functions in a Sturm-Liouville problem.

- (a) Determine the *bounded* eigenfunctions y_j and eigenvalues λ_j of the following singular Sturm-Liouville problem on $0 \leq x \leq 1$:

$$-(xy'(x))' = \lambda xy(x), \quad y(1) = 0.$$

Hint: Use a change of variables of the form $r = \beta x$.

- (b) Use (a) to obtain the eigenfunction expansion for the *bounded* solution of the following inhomogeneous problem on $0 \leq x \leq 1$:

$$(xy'(x))' = x, \quad y(1) = 0.$$

Leave the coefficients c_k in your final answer in terms of integrals containing Bessel functions.

5. Legendre functions and associated Legendre functions. Consider Legendre's equation

$$(1-x^2)y''(x) - 2xy'(x) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)y(x) = 0,$$

and let $P_\ell^m(x)$ denote the solution for integers $0 \leq m \leq \ell$. Show that

$$\int_{-1}^1 P_k^m(x) P_\ell^m(x) dx = \begin{cases} 0 & \text{if } \ell \neq k \\ \frac{2}{(2k+1)} \frac{(k+m)!}{(k-m)!} & \text{if } \ell = k. \end{cases}$$

[You may use without proof Rodrigues' formula given in lectures, and also the identity

$$\int_{-1}^1 (1-x^2)^\ell dx = \frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!},$$

or for extra fun try to show this as well...]