

Problem Sheet 4

1. Order relations.

(a) Determine the order as $\epsilon \rightarrow 0$ of

$$\frac{\sqrt{\epsilon}}{1 - \cos \epsilon}.$$

(b) Obtain an asymptotic expansion of $\exp(\tan \epsilon)$ in integer powers of ϵ up to order $O(\epsilon^4)$.

(c) Show that $\log \epsilon = o(\epsilon^{-p})$ as $\epsilon \rightarrow 0^+$ for all $p > 0$.

2. Roots. Find expansions for all roots of the equations below as $\epsilon \rightarrow 0$ with two nonzero terms in each expansion:

(a) $\epsilon x^3 - x + 1 = 0,$

(b) $\tan x = \frac{\epsilon}{x}.$

[*Hint: Roots are near $n\pi$ for integers n . The root near zero must be treated separately, and requires the balance $x = O(\epsilon^{1/2})$ (can you see why?).]*

3. Regular perturbation. Find the first two terms in an asymptotic expansion in powers of the small parameter ϵ of the solution of

$$xy'(x) + y(x) = \epsilon y(x)^{1/2}, \quad x > 1, \quad y(1) = 1.$$

Explain why the expansion is not valid as $x \rightarrow \infty$. What form of rescaling would be necessary to examine the behaviour for large x ?

[*You do not need to carry out the resulting analysis.*]

4. Inner and outer expansions. Find inner and outer expansions, correct up to and including terms of $O(\epsilon)$, for the function

$$f(x; \epsilon) = \frac{e^{-x/\epsilon}}{x} + \frac{\sin x}{x} - \coth x,$$

for $x > 0$ and $0 < \epsilon \ll 1$. Compare the inner and outer approximations with the exact function f by plotting all three on the same graph for various small values of ϵ .

5. Singular perturbation. Use matched asymptotic expansions to find leading-order outer and inner solutions to the boundary value problem

$$\begin{aligned} \epsilon y''(x) + xy'(x) + y(x) &= 0, & 1 < x < 2, \\ y(1) &= 0, & y(2) &= 1, \end{aligned}$$

where $0 < \epsilon \ll 1$. Form the leading-order composite solution.

Explain briefly why there is a boundary layer at $x = 1$ and not at $x = 2$.

How might the analysis change if ϵ were small and *negative*? [*You do not need to find the solution in this case.*]

6. Singular perturbation 2. Construct leading-order inner and outer solutions to

$$\begin{aligned} \epsilon u''(x) + u'(x) &= \frac{u(x) + u(x)^3}{1 + 3u(x)^2}, & 0 < x < 1, \\ u(0) &= 0, & u(1) &= 1 \end{aligned}$$

where $0 < \epsilon \ll 1$. [*You will only be able to determine the outer solution implicitly.*]