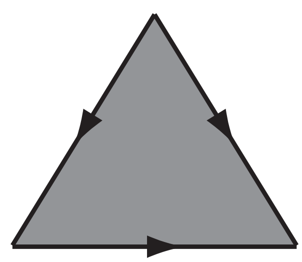
PART A TOPOLOGY
HT 2019
EXERCISE SHEET 4

Vacation sheet

Simplicial complexes

Exercise 1. (1) Show that the following space (the *'Dunce hat'*) can be triangulated.



(2) Show that the following subspace of $\mathbb{R}^2$ cannot be triangulated:

$$\{(x, y) : 0 \leq y \leq 1, \text{ and } x = 0 \text{ or } 1/n, \text{ for some } n \in \mathbb{N}\} \cup ([0, 1] \times \{0\}).$$

[*Hint*: It is helpful to show that, for any finite simplicial complex K, any point $x \in |K|$ and any open set U containing x, there is a connected open set V such that $x \in V \subseteq U$.]

Exercise 2. Let K be a simplicial complex (that need not be finite). Prove that $|K|$ is Hausdorff.

[*Hint*: Recall that a subset of $|K|$ is open if it intersects every simplex in an open set. Note also that the standard simplex has a natural metric as a subset of $\mathbb{R}^n$].

Surfaces

Exercise 3. Let X_1, X_2 be disjoint copies of $\mathbb{R}^2$. We define an equivalence relation $\sim$ on $Y = X_1 \coprod X_2$ by: $(x_1, y_1) \in X_1$ is equivalent to $(x_2, y_2) \in X_2$ if and only if $x_1 = x_2, y_1 = y_2$ and $(x_1, y_1), (x_2, y_2)$ are not equal to $(0, 0)$. Show that every point in $Y/\!\sim$ is contained in an open set homeomorphic to an open subset of $\mathbb{R}^2$ but $Y/\!\sim$ is not a surface.

Exercise 4. Find an example of a connected, finite, simplicial complex K that is *not* a closed combinatorial surface, but that satisfies the following three conditions:

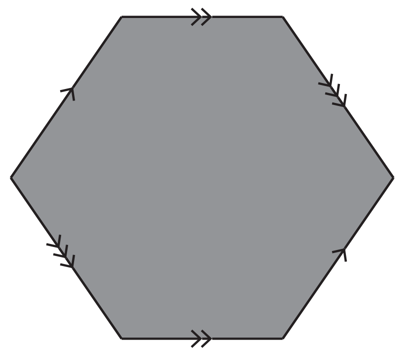
(1) It contains only 0-simplices, 1-simplices and 2-simplices.
(2) Every 1-simplex is a face of precisely two 2-simplices.
(3) Every point of $|K|$ lies in a 2-simplex.

Exercise 5. A *simple closed curve* C in a space X is the image of a continuous injection $S^1 \to X$. Find simple closed curves C_1, C_2 and C_3 in the Klein bottle K such that

(1) $K \backslash C_1$ has one component, which is homeomorphic to an open annulus $S^1 \times (0, 1)$.
(2) $K \backslash C_2$ has one component, which is homeomorphic to an open Möbius band.
(3) $K \backslash C_3$ has two components, each of which is homeomorphic to an open Möbius band.

[An *open Möbius band* is the space obtained from $[0, 1] \times (0, 1)$ by identifying $(0, y)$ with $(1, 1 - y)$ for each $y \in (0, 1)$.]

Exercise 6. The following polygon with side identifications is homeomorphic to which surface?



Exercise 7. Suppose that the sphere $\mathbb{S}^2$ is given the structure of a closed combinatorial surface. Let C be a subcomplex that is a simplicial circle. Suppose that $\mathbb{S}^2 \backslash C$ has two components. Indeed, suppose that this is true for every simplicial circle in $\mathbb{S}^2$. Let E be one of these components. [In fact, $\mathbb{S}^2 \backslash C$ must have 2 components, but we will not attempt to prove this.]

Our aim is to show that $\overline{E}$ is homeomorphic to a disc. This is a version of the *Jordan curve theorem*.

We'll prove this by induction on the number of 2-simplices in $\overline{E}$. Our actual inductive hypothesis is: *There is a homeomorphism from* $\overline{E}$ *to* $\mathbb{D}^2$*, which takes* C *to the boundary circle* $\partial\mathbb{D}^2$.

(1) Let σ_1 be a 1-simplex in C. Since $\mathbb{S}^2$ is a closed combinatorial surface, σ_1 is adjacent to two 2-simplices. Show that precisely one of these 2-simplices lies in $\overline{E}$. Call this 2-simplex σ_2.

(2) Start the induction by showing that if $\overline{E}$ contains at most one 2-simplex, then $\overline{E} = \sigma_2$.

(3) Let v be the vertex of σ_2 not lying in σ_1. Let's suppose that v does not lie in C. Show how to construct a subcomplex C' of $\mathbb{S}^2$, that is a simplicial circle, and that has the following properties:
- $\mathbb{S}^2 \backslash C'$ has two components;
- one of these components F is a subset of E;
- $\overline{F}$ contains fewer 2-simplices than $\overline{E}$.

Show in this case that there is a homeomorphism from $\overline{E}$ to $\mathbb{D}^2$, which takes C to the boundary circle $\partial\mathbb{D}^2$.

(4) Suppose now that v lies in C. How do we complete the proof in this case?

[The actual Jordan curve theorem is rather stronger than this. It deals with simple closed curves C in $\mathbb{S}^2$, which need to be simplicial. It states that $\mathbb{S}^2 \backslash C$ has two components, and that, for each component E of $\mathbb{S}^2 \backslash C$, the closure of E is homeomorphic to $\mathbb{D}^2$, with the homeomorphism taking C to $\partial\mathbb{D}^2$.]