Part A Integration: HT 2020

Problem Sheet 1: Lebesgue measure

An asterisk before the number of a question, or a part of a question, indicates that it is optional. Such questions may cover proofs omitted from the lectures or other topics related to the course, and some may be a bit harder. Strong students should be encouraged to do some of them, but I would expect only a few to attempt all parts of all questions.

1. Let $f_n(x) = n^2 x^n (1-x)$ $(0 \le x \le 1)$. Show that
   (i) $\lim_{n\to\infty} f_n(x)\,dx = 0$ for each $x \in [0,1]$.
   (ii) $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$.

2. Show that $\displaystyle\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3}\,dx \right) dy = -\frac{1}{2}$.

   Deduce that $\displaystyle\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3}\,dy \right) dx \ne \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3}\,dx \right) dy.$

3. (a) Let $E = \mathbb{Q} \cap [0,1]$. Show that there exists a sequence $(x_n)_{n\ge1}$ in $[0,1]$ such that the sets $E + x_n := \{y + x_n : y \in E\}$ $(n = 1, 2, \dots)$ are disjoint. Show that

$$0 \le \sum_{n=1}^{k} \chi_E(x - x_n) \le \chi_{[0,2]}(x) \qquad (x \in \mathbb{R}, k \in \mathbb{N}).$$

   (b) Let V be a vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, and $\phi : V \to \mathbb{R}$ be a linear functional with the following properties:
   (i) For any bounded interval $I \subseteq \mathbb{R}$ with endpoints a and b, $\chi_I \in V$ and $\phi(\chi_I) = b - a$.
   (ii) If $f \in V$ and $f(x) \ge 0$ for all $x \in \mathbb{R}$, then $\phi(f) \ge 0$.
   (iii) If $f \in V$, $a \in \mathbb{R}$ and $f_a(x) = f(x-a)$, then $f_a \in V$ and $\phi(f_a) = \phi(f)$.

   If $\chi_E \in V$, show that $\phi(\chi_E) = 0$.

4. Find $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ when
   (i) $a_n = \exp(-\cos n)$,
   (ii) $a_n = \exp\left(n \sin\left(\frac{n\pi}{2}\right)\right) + \exp\left(\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right)$,
   (iii) $a_n = \cosh\left(n \sin\left(\left(\frac{n^2+1}{n}\right)\frac{\pi}{2}\right)\right)$.

5. Let (a_n) and (b_n) be bounded real sequences. Prove that
   (i) If $a_n \le b_n$ for all n then $\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$.
   (ii) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
   *(iii) There is a subsequence $(a_{n_r})_{r\ge1}$ of (a_n) such that $\displaystyle\lim_{r\to\infty} a_{n_r} = \limsup_{n\to\infty} a_n$.
   *(iv) If $(a_{k_r})_{r\ge1}$ is any convergent subsequence of (a_n), then $\displaystyle\lim_{r\to\infty} a_{k_r} \le \limsup_{n\to\infty} a_n$.

6. Let C be the Cantor set. Explain, in as much detail as you think is appropriate, why
$$C = \left\{ \sum_{n=1}^{\infty} a_n 3^{-n} : a_n = 0 \text{ or } 2 \right\}.$$
Prove that C is uncountable, for example by either (or both) of the following methods:
   (a) adapting Cantor's proof, via decimal expansions, that $[0,1]$ is uncountable,
   (b) constructing a surjection of C onto $[0,1]$—think about binary expansions in $[0,1]$.

   *Prove that $C + C = [0,2]$ and deduce that C is uncountable.

7. Show that the set of all real numbers which have a decimal expansion not containing the digit 4 is null. [Consider first numbers between 0 and 1.]

   Show that if A is null and B is countable, then $A + B$ is null.

   Show that if A is null and $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, then $f(A)$ is null. [Consider first the case when $A \subseteq [0,1]$ and use the fact that f' is bounded on $[0,1]$.]

8. Let A, B and A_n be subsets of $\mathbb{R}$, $x, \alpha \in \mathbb{R}$. Prove the following
   (i) $m^*(A + x) = m^*(A)$,
   (ii) $m^*(\alpha A) = |\alpha| m^*(A)$,
   (iii) $m^*(A \cup B) \leq m^*(A) + m^*(B)$,
   (iv) $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

*9. Prove the following:
   (i) Any null set is (Lebesgue) measurable.
   (ii) Any interval is measurable.
   (iii) If E and F are measurable and $x, \alpha \in \mathbb{R}$, then $E + x$, αE and $E \cup F$ are measurable.
   (iv) If E_n are disjoint measurable subsets of $\mathbb{R}$, then $\bigcup_{n=1}^{\infty} E_n$ is measurable and $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*(E_n)$.

*10. Let G be an open subset of $\mathbb{R}$. For $x, y \in G$, let $I_{x,y}$ be the closed (or open, if you prefer) interval between x and y, so $I_{x,x} = \{x\}$ (or $\emptyset$). Define a relation $\sim$ on G by $x \sim y$ if and only if $I_{x,y} \subseteq G$.
   (i) Show that $\sim$ is an equivalence relation on G.
   (ii) Show that each equivalence class is an open interval. [To show that A is an interval, it is sufficient to check that, if $x, y \in A$ then $I_{x,y} \subseteq A$.]
   (iii) Show that there are (at most) countably many equivalence classes. [Think about rational numbers.]
   (iv) Deduce that G is the union of (at most) countably many, disjoint open intervals.