## Part A Integration: HT 2020

**Problem Sheet 2**: Measurable functions, Lebesgue integral

1. Let $(\omega_n)$ be a sequence of non-negative real numbers. For a subset $E$ of $\mathbb{N}$, let

$$\mu_\omega(E) = \sum_{n \in E} \omega_n.$$

Show that $\mu_\omega$ is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

Now let $\nu$ be any measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, and define $\omega_n = \nu(\{n\})$. Show that $\nu(E) = \mu_\omega(E)$ for all subsets $E$ of $\mathbb{N}$.

2. Let $(\Omega, \mathcal{F}, \mu)$ be any measure space. If $(A_n)$ is a decreasing sequence of sets in $\mathcal{F}$ and $\mu(A_1) < \infty$, prove that

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Is this still true if $\mu(A_1) = \infty$?

3. Let $b \in \mathbb{R}$. Show that $(-\infty, b) = \bigcup_{n=1}^{\infty} \left(\mathbb{R} \setminus (b - \frac{1}{n}, \infty)\right)$.

Deduce that if $\mathcal{F}$ is a $\sigma$-algebra on $\mathbb{R}$ containing the intervals $(a, \infty)$ for each $a \in \mathbb{R}$, then $\mathcal{F}$ contains all open intervals, and hence all open subsets of $\mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that

$$\mathcal{G} := \left\{G \subseteq \mathbb{R} : f^{-1}(G) \in \mathcal{M}_{\text{Leb}}\right\}$$

is a $\sigma$-algebra. Deduce that if $f^{-1}(a, \infty) \in \mathcal{M}_{\text{Leb}}$ for every $a$, then $f^{-1}(G) \in \mathcal{M}_{\text{Leb}}$ for every $G \in \mathcal{M}_{\text{Bor}}$.

4. Let $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be a non-empty family of $\sigma$-algebras on the same set $\Omega$. Show that $\bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a $\sigma$-algebra.

By considering the family of all $\sigma$-algebras containing $\mathcal{B}$, deduce that if $\mathcal{B}$ is any subset of $\mathcal{P}(\Omega)$, there is a unique $\sigma$-algebra $\mathcal{F}_\mathcal{B}$ such that

(i) $\mathcal{B} \subseteq \mathcal{F}_\mathcal{B}$;

(ii) If $\mathcal{G}$ is a $\sigma$-algebra on $\Omega$ and $\mathcal{B} \subseteq \mathcal{G}$, then $\mathcal{F}_\mathcal{B} \subseteq \mathcal{G}$.

*5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\Omega_*$ be a set, and $f : \Omega \to \Omega_*$ be a function. Let

$$f_*(\mathcal{F}) = \{G \subseteq \Omega_* : f^{-1}(G) \in \mathcal{F}\}, \quad (f_*\mu)(G) = \mu(f^{-1}(G)).$$

Show that $(\Omega_*, f_*(\mathcal{F}), f_*\mu)$ is a measure space.

Now let $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{M}_{\text{Bor}}, m)$, and $\Omega_* = \mathbb{R}$. Determine $f_*(\mathcal{M}_{\text{Bor}})$ and $f_*m$ when

(i) $f(x) = \tan x$ if $\cos x \neq 0$, and $f(x) = 0$ if $\cos x = 0$,

(ii) $f(x) = \arctan x$ (taking values in $(-\pi/2, \pi/2)$).

6. *(a) Let $I$ be an interval of positive length, let $a \in I$, $f, g : I \to \mathbb{R}$ be functions such that $f(x) = g(x)$ a.e., and suppose that $f$ and $g$ are continuous at $a$. Show that $f(a) = g(a)$.

   (b) Is $\chi_{\mathbb{Q}}$ continuous a.e.? Does there exist a continuous function $g$ such that $\chi_{\mathbb{Q}} = g$ a.e.?

   (c) Is $\chi_{(0,\infty)}$ continuous a.e.? Does there exist a continuous function $g$ such that $\chi_{(0,\infty)} = g$ a.e.? [*Use (a).*]

7. Let $f, g$ be measurable functions from $\mathbb{R}$ to $\mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be continuous. Recall from lectures that $f + g$ and $h \circ f$ are measurable. Prove that the following functions are measurable. [*Complicated constructions are not required. Everything can be quickly deduced from the information from lectures recalled above, plus a couple of simple formulae.*]
   (i) $f^2 : x \mapsto f(x)^2$,
   (ii) $fg : x \mapsto f(x)g(x)$,
   (iii) $|f| : x \mapsto |f(x)|$,
   (iv) $\max(f, g) : x \mapsto \max(f(x), g(x))$.

*8. Suppose that $g$ is a measurable function and $f = g$ a.e. Show that $f$ is measurable.

   Suppose that $f$ is continuous a.e. Show that there is a sequence of step functions $(\phi_n)$ such that $f = \lim_{n\to\infty} \phi_n$ a.e. Deduce that $f$ is measurable.

9. Let $f : \mathbb{R} \to [-\infty, \infty]$ be an integrable function, and let $\alpha > 0$. Show that

$$m(\{x : |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \int |f|.$$

   Deduce that
   (i) $f(x) \in \mathbb{R}$ a.e.
   (ii) If $\int |f| = 0$, then $f(x) = 0$ a.e.

10. In each of the following cases, state whether the function $f$ is Lebesgue integrable over the interval $I$. Justify your answers, *and calculate $\int_I f$ in those cases where this is feasible.
    (i) $I = \mathbb{R}$, $f(x) = x$ if $x$ is rational, $f(x) = 0$ if $x$ is irrational,
    (ii) $I = (0, \pi/2)$, $f(x) = \tan x$,
    (iii) $I = [1, \infty)$, $f(x) = (-1)^n/n$ if $n \leq x < n + 1, n = 1, 2, 3, \ldots,$
    (iv) $I = (0, 1]$, $f(x) = \sin(1/x)$,
    (v) $I = [0, \infty)$, $f(x) = x^n e^{-x}$ where $n$ is a positive integer,
    (vi) $I = (0, \infty)$, $f(x) = (\log x)e^{-x}$,
    *(vii) $I = [1, \infty)$, $f(x) = x^\alpha \log x$ where $\alpha \in \mathbb{R}$,
    *(viii) $I = (0, \pi)$, $f(x) = (\operatorname{cosec} x)^{1/2}$,
    *(ix) $I = (0, \infty)$, $f(x) = (1 + x)^{-1} \cos x$,
    *(x) $I = [1, \infty)$, $f(x) = \sin(1/x)$.