Part A Integration

Problem Sheet 3: Convergence theorems and consequences

Note: *Answers to this problem sheet should, if possible, be carefully justified by reference to theorems in the lectures and by showing that the conditions of those theorems are satisfied.*

1. Let $\alpha \in (0, \infty)$. Show that $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ exists as a Lebesgue integral, and that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

2. We know that $\dfrac{\sin x}{x}$ is not integrable over $(1, \infty)$. Deduce, or prove otherwise, that neither of the following functions is integrable over the given intervals:

$$\text{(i)} \quad \frac{\sin\left(x^2\right)}{x} \quad \text{over} \quad (1, \infty), \qquad \text{(ii)} \quad \frac{1}{x} \sin \frac{1}{x^2} \quad \text{over} \quad (0, 1).$$

3. By comparing terms in binomial expansions, or otherwise, show that $\left(1 + \frac{x}{n}\right)^n \le \left(1 + \frac{x}{n+1}\right)^{n+1}$ for $n \ge 2$, $x \ge 0$. Use the MCT to deduce that

$$\lim_{n\to\infty} \int_1^2 \left(1 + \frac{x}{n}\right)^{-n} dx = e^{-1} - e^{-2}.$$

*4. Use the binomial expansion of $(1 - x)^{-k}$ to show that $\left(1 - \frac{u}{n}\right)^{-n} \ge \left(1 - \frac{u}{n+1}\right)^{-(n+1)}$ for $n \ge 2$ and $0 \le u \le n$. Hence, or otherwise, prove that

$$\lim_{n\to\infty} n^\alpha \int_0^1 x^{\alpha-1} e^{-n\beta x} (1 - x)^n\, dx = (\beta + 1)^{-\alpha}\Gamma(\alpha)$$

where $\alpha > 0$, $\beta > -1$, and $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1}\, du$.

5. Show that $0 \le \frac{x}{1+x^\alpha} \le 1$ for $\alpha > 1$, $x \ge 0$, and deduce that

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{nx \sin x}{1 + n^\alpha x^\alpha}\, dx = 0.$$

*6. Prove that $\displaystyle \lim_{n\to\infty} \int_0^{n^2} n\left(\sin \frac{x}{n}\right) e^{-x^2}\, dx = \frac{1}{2}.$

7. Let $\alpha > -1$. Show that $x^\alpha \log x$ is integrable over $(0,1)$, and
$$\int_0^1 x^\alpha \log x \, dx = -(1+\alpha)^{-2}.$$
Deduce that for $\beta > -1$, $x^\beta(1-x)^{-1}\log x$ is integrable over $(0,1)$, and
$$\int_0^1 x^\beta (1-x)^{-1} \log x \, dx = -\sum_{n=1}^{\infty} (n+\beta)^{-2}.$$

8. Show that for $n > 0$, $e^{-nx}\sin x$ is integrable over $[0,\infty)$, and
$$\int_0^\infty e^{-nx} \sin x \, dx = \frac{1}{1+n^2}.$$
Deduce that for $0 \le a \le 1$, $(e^x - a)^{-1}\sin x$ is integrable over $[0,\infty)$, and
$$\int_0^\infty (e^x - a)^{-1} \sin x \, dx = \sum_{n=1}^{\infty} \frac{a^{n-1}}{1+n^2}.$$

*9. Let $a \in (0,1)$. Prove that $\dfrac{e^{-ax} - e^{(a-1)x}}{1 - e^{-x}}$ is an integrable function on $\mathbb{R}$, and that
$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-ax} - e^{(a-1)x}}{1-e^{-x}} \, dx = \frac{1}{a} + \frac{1}{a-1} + \frac{1}{a+1} + \frac{1}{a-2} + \frac{1}{a+2} + \frac{1}{a-3} + \dots$$

10. Prove that $\displaystyle\int_0^\infty \cos x \arctan(\lambda x)\, e^{-x} \, dx \to \frac{\pi}{4}$ as $\lambda \to \infty$.

11. Let $J_0(x) = \displaystyle\frac{2}{\pi}\int_0^{\pi/2} \cos(x\cos\theta)\, d\theta$. Show that J_0 is differentiable on $\mathbb{R}$.

*Show that the function Γ, as defined in Q.1, is differentiable on $(0,\infty)$.

12. Let $f(x,y) = y^3 e^{-y^2 x}$, $F(y) = \displaystyle\int_0^\infty f(x,y)\, dx$. Calculate $F'(0)$ and $\displaystyle\int_0^\infty \frac{\partial f}{\partial y}(x,0)\, dx$.
How do your answers relate to the theorem about differentiating through integrals?

13. By differentiating through the integral sign, evaluate the following integrals:

(i) $\displaystyle\int_0^\infty \frac{e^{-x}\sin tx}{x}\, dx$,

*(ii) $\displaystyle\int_0^{\pi/2} \log(a^2\cos^2 x + b^2 \sin^2 x)\, dx$, where $a, b > 0$.

*14. Let $K(t) = \displaystyle\int_1^\infty \frac{\cos(tx)}{x^2}\, dx$. Show carefully that $K'(t) = \dfrac{K(t) - \cos t}{t}$ for $t > 0$.