

Part A Integration

Problem Sheet 4: Fubini's Theorem, L^p -spaces

- Evaluate $\int_0^1 \left(\int_0^x e^{-y} dy \right) dx$ and $\int_0^1 \left(\int_0^{x-x^2} (x+y) dy \right) dx$
 - directly;
 - by reversing the order of integration.
- In each of the following cases, is f integrable over the given region? [Give careful justification.]
 - $f(x, y) = e^{-xy}$ over $[0, \infty) \times [0, \infty)$;
 - $f(x, y) = e^{-xy}$ over $\{(x, y) : 0 < x < y < x + x^2\}$;
 - $f(x, y) = \frac{(\sin x)(\sin y)}{x^2 + y^2}$ over $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

3. [Applications of Tonelli or Fubini should be carefully justified.]

(a) Let $J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \cos \theta) d\theta$. Show that $\int_0^\infty J_0(x) e^{-ax} dx = \frac{1}{\sqrt{1+a^2}}$ if $a > 0$.

(b) Take $b > a > 1$. By considering x^{-y} over $(1, \infty) \times (a, b)$, show that $\int_1^\infty \frac{x^{-a} - x^{-b}}{\log x} dx$ exists, and find its value.

4. Let $f \in \mathcal{L}^1(\mathbb{R})$ be non-negative with $\int_{-\infty}^\infty f(x) dx = 1$, and let $F(x) = \int_{-\infty}^x f(y) dy$. Assume that $xf(x) \in \mathcal{L}^1(\mathbb{R})$. Use Fubini's Theorem to prove that

$$\int_0^\infty (1 - F(x)) dx = \int_0^\infty xf(x) dx, \quad \int_{-\infty}^0 F(x) dx = - \int_{-\infty}^0 xf(x) dx.$$

Now let g be a bounded measurable function, and let

$$G(y) = \int_{\{g(x) \leq y\}} f(x) dx.$$

Prove that

$$\int_0^\infty (1 - G(y) - G(-y)) dy = \int_{-\infty}^\infty f(x)g(x) dx.$$

[*Remark (not a hint):* Imagine that f is the probability density function of a random variable X . The first part of the question then says that $\mathbb{E}(X) = \int_0^\infty (\mathbb{P}[X > x] - \mathbb{P}[X \leq -x]) dx$. This formula holds for all random variables (discrete, continuous, etc) with $\mathbb{E}(|X|) < \infty$. In particular it holds for $g(X)$. Then the last part proves that $\mathbb{E}[g(X)] = \int_{-\infty}^\infty f(x)g(x) dx$, a fact sometimes known as the Law of the Unconscious Statistician.]

5. (a) Let $\alpha > 1$ and $f(x, y) = (x^2 + y^2)^{-\alpha}$ and $g(x, y) = (1 + x^2 + y^2)^{-\alpha}$. Show that f is integrable over $[1, \infty) \times [0, \infty)$ [Hint: Change of variables $y = ux$ may help]. Deduce that f is integrable over $[0, 1] \times [1, \infty)$, and that g is integrable over \mathbb{R}^2 .

(b) Use polar coordinates to show that g is integrable over \mathbb{R}^2 .

6. For $p > 0$, calculate $\|f\|_p$ when f is (i) $\chi_{(0,1)}$, (ii) $\chi_{(1,2)}$, (iii) $\chi_{(0,2)}$.

Now assume that $0 < p < 1$. Is $\|\cdot\|_p$ a norm on L^p ?

For $f, g \in L^p(\mathbb{R})$, let $d_p(f, g) = \int |f - g|^p$. Show that d_p is a metric on $L^p(\mathbb{R})$.

7. Consider the relation \sim on the space of measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ given by:

$$f \sim g \iff f = g \text{ a.e.}$$

State which properties of null sets are used to prove each of the following true statements (f, g, h , etc are measurable functions):

(i) $f \sim f$,

(ii) $f \sim g \implies g \sim f$,

(iii) $f \sim g, g \sim h \implies f \sim h$,

(iv) If $f_n \sim g_n$ for all $n \in \mathbb{N}$, then $\sup f_n \sim \sup g_n$,

(v) If $f \sim g$, then $h \circ f \sim h \circ g$.

*Give an example where h is injective, $f \sim g$, but $f \circ h \not\sim g \circ h$.

8. Let $p > 1$. Give examples of sequences (f_n) and (g_n) in $L^p(0, 1)$ such that

(i) $\lim_{n \rightarrow \infty} f_n(x) = 0$ a.e. but $\lim_{n \rightarrow \infty} \|f_n\|_p \neq 0$;

(ii) $\lim_{n \rightarrow \infty} \|g_n\|_p = 0$ but $\lim_{n \rightarrow \infty} g_n(x)$ does not exist for any $x \in (0, 1)$.

For each $\varepsilon > 0$ find a measurable subset E_ε of $[0, 1]$ such that $m(E_\varepsilon) < \varepsilon$ and $f_n(x) \rightarrow 0$ uniformly on $[0, 1] \setminus E_\varepsilon$.

Find a subsequence (g_{n_r}) such that $\lim_{r \rightarrow \infty} g_{n_r}(x) = 0$ a.e.

*9. A function $g : [0, \infty) \rightarrow \mathbb{R}$ is *convex* if

$$g(x) = \sup\{\alpha x + \beta : \alpha y + \beta \leq g(y) \text{ for all } y \in [0, \infty)\}.$$

If g is continuous on $[0, \infty)$ with non-negative second derivative on $(0, \infty)$, then g is convex.

Let $f : [0, 1] \rightarrow [0, \infty)$ be bounded, measurable, and $M_n = \int_0^1 f^n dx = \|f\|_{L^n}^n$. Show that

(i) $g\left(\int_0^1 f(x) dx\right) \leq \int_0^1 g(f(x)) dx$ for every convex function g ;

(ii) $M_n^2 \leq M_{n+1}M_{n-1}$;

(iii) $\|f\|_{L^n} \leq M_{n+1}/M_n \leq \|f\|_{L^\infty}$;

(iv) $\lim_{n \rightarrow \infty} M_{n+1}/M_n = \|f\|_{L^\infty}$.