# Definitions of measurable sets, and regularity of Lebesgue measure

**Three equivalent definitions**

As mentioned in the course, there are numerous ways of defining Lebesgue measurable subsets of $\mathbb{R}$. The definition given in the lectures, and also in Zhongmin Qian's lecture notes, and in the book of Capinski & Kopp, is that a subset $E$ of $\mathbb{R}$ is Lebesgue measurable if

$$(0.1) \qquad m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

for all subsets $A$ of $\mathbb{R}$. We write $E \in \mathcal{M}_{\mathrm{Leb}}$ when this holds.

An alternative definition, given in the book of Stein & Shakarchi and elsewhere, is that $E$ is measurable if, for every $\varepsilon > 0$, there exists an open set $U$ such that $E \subseteq U$ and $m^*(U \setminus E) < \varepsilon$. We write $E \in \mathcal{M}_{\mathrm{SS}}$ when this holds. This definition can, in principle, be checked by identifying sets $U$ for countably many $\varepsilon > 0$.

There is a third definition in Garling's book. He defines outer measure (denoted by $\lambda^*$ in his case) in essentially the same way as in our course, but he also defines the *size* of a compact subset $K$ of $\mathbb{R}$ to be $s(K) = m^*(U) - m^*(U \setminus K)$ where $U$ is any bounded open subset of $\mathbb{R}$ containing $K$. Garling shows that $s(K)$ is independent of the choice of $U$ and then defines the inner measure of $E \subseteq \mathbb{R}$ to be

$$m_*(E) = \sup \left\{ s(K) : K \text{ compact}, K \subseteq E \right\}.$$

Then he defines $E$ to be measurable if $m^*(E_n) = m_*(E_n)$ for all $n \geq 1$, where $E_n = E \cap [-n, n]$. We write $E \in \mathcal{M}_{\mathrm{Gar}}$ when this holds. It is sufficient that $m^*(E) = m_*(E)$ if the common value is finite, but not if it is infinite.

We now show that these three definitions are equivalent, using facts from our course. We may use the notation $m(E)$ instead of $m^*(E)$ when $E \in \mathcal{M}_{\mathrm{Leb}}$. For example, it is clear from our course that $s(K) = m(U) - (m(U) - m(K)) = m(K)$ for any compact set $K$ and bounded open $U \supseteq K$.

First assume that $E \in \mathcal{M}_{\mathrm{Leb}}$. By definition of $m^*$ using open intervals, there exist open $U_n$ such that $E_n \subseteq U_n$ and $m^*(U_n) < m^*(E_n) + \varepsilon 2^{-n}$. Let $U = \bigcup_n U_n$. Then $U$ is open, $E \subseteq U$ and $U \setminus E \subseteq \bigcup_n (U_n \setminus E_n)$. Hence

$$m^*(U \setminus E) \leq \sum_n m^* (U_n \setminus E_n) = \sum_n (m(U_n) - m(E_n)) < \varepsilon.$$

So $E \in \mathcal{M}_{\mathrm{SS}}$.

On the other hand, assume that $E \in \mathcal{M}_{\mathrm{SS}}$. Then for each $n$, there exists an open set $U_n$ such that $E \subseteq U_n$ and $m^*(U_n \setminus E) < n^{-1}$. Let $G = \bigcap U_n$, so $G$ is (Borel) measurable, $E \subseteq G$ and $m^*(G \setminus E) = 0$. This implies that $G$ and $G \setminus E$ are Lebesgue measurable, and hence so is $(\mathbb{R} \setminus G) \cup (G \setminus E) = \mathbb{R} \setminus E$. So $E \in \mathcal{M}_{\mathrm{Leb}}$.

Now suppose that $E \in \mathcal{M}_{\mathrm{Leb}} = \mathcal{M}_{\mathrm{SS}}$. Let $n \geq 1$ and $\varepsilon > 0$. Since $E_n \in \mathcal{M}_{\mathrm{SS}}$ there exists an open set $U$ such that $E_n \subseteq U$ and $m^*(U \setminus E_n) < \varepsilon$, so $m(U) < m(E_n) + \varepsilon$. Replacing $E_n$ by $[-n, n] \setminus E_n$, there exists an open set $V$ such that $[-n, n] \setminus E_n \subseteq V$ and $m(V) < m([-n, n] \setminus E_n) + \varepsilon$, so $m(E_n \cap V) < \varepsilon$. Let $K = [-n, n] \cap (\mathbb{R} \setminus V)$, so

$K$ is compact, $E_n = K \cup (E_n \cap V)$, $K \subseteq E_n$ and $m(E_n) < m(K) + \varepsilon$. It follows that $m^*(E_n) = m(E_n) = m_*(E_n)$.

If $E \in \mathcal{M}_{\mathrm{Gar}}$, then, for each $n$, there exist compact $K_n$ and open $U_n$ such that $K_n \subseteq E_n \subseteq U_n$ and $m(U_n \setminus K_n) = m(U_n) - m(K_n) < 2^{-n}$. Let

$$A = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} K_n, \qquad B = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} U_n.$$

Then $A$ and $B$ are (Borel) measurable, $A \subseteq E \subseteq B$ and $m(B \setminus A) = 0$ since $B \setminus A \subseteq \bigcup_{n \geq m}(U_n \setminus K_n)$ for any $m$. Thus $E$ is the union of a Borel set $A$ and a null set $E \setminus A$, so $E \in \mathcal{M}_{\mathrm{Leb}}$.

## Regularity of Lebesgue measure

As a consequence of the arguments above, the following hold for any Lebesgue measurable set $E$:

$$
\begin{aligned}
m(E) &= \inf\{m(U) : U \text{ open}, E \subseteq U\} && \text{(outer regularity)}, \\
&= \sup\{m(K) : K \text{ compact}, K \subseteq E\} && \text{(inner regularity)}.
\end{aligned}
$$

Outer regularity is almost immediate from the definition of $m^*$. Inner regularity follows from $m(E) = \sup_n m(E_n) = \sup_n m_*(E_n)$.

## Example 2.4

For the set $A$ described in Example 2.4 of the course, $m_*(A) = 0$. It is impossible to say what is the value of $m^*(A)$ because $A$ is not specified explicitly. All we can say is that $0 < m^*(A) \leq 1$.

## Priestley definition

Priestley defines a set $E \subseteq \mathbb{R}$ to be measurable if $\chi_E$ is the limit a.e. of a sequence of step functions $\psi_n$. This is equivalent to our definition by the (unproven) Theorem 3.10 in the lecture notes.

## Etheridge definition

The property (0.1) is a convenient definition for deducing properties of measurable sets, but it is impossible to check it directly for individual sets $E$, because it involves all subsets of $\mathbb{R}$, including any which might be non-measurable. When Alison Etheridge gave the course in 2008, 2010 and 2011, she defined measurable sets by requiring (0.1) to hold not for arbitrary subsets $A$, but only for bounded intervals. She wrote that the two definitions are equivalent, but *that is not a very easy thing to show.*