$\mathcal{M}_{\mathbf{Bor}} \neq \mathcal{M}_{\mathbf{Leb}}$

Here, we describe two ways of seeing that there exist Lebesgue measurable subsets of $\mathbb{R}$ which are not Borel measurable (Prop 3.3). Neither description is completely explicit.

1. Let A be a subset of $[0,1]$ which is not Lebesgue measurable (the existence of this relies on the Axiom of Choice), and let C be the Cantor set. Define a function $\Psi : [0,1] \to C$ as follows:

Any $x \in (0,1]$ has a unique non-terminating binary expansion $x = \sum_{n=1}^{\infty} a_n 2^{-n}$, where $a_n = 0$ or 1 for each n and $a_n = 1$ for infinitely many n. Define

$$\Psi(x) = \sum_{n=1}^{\infty} (2a_n) 3^{-n} \in C.$$

Define $\Psi(0) = 0$.

Now $\Psi(A) \subseteq C$, so $\Psi(A)$ is null and hence Lebesgue measurable. Moreover, Ψ is strictly monotonic increasing and hence measurable (Example 3.5(4)). Also, $\Psi^{-1}(\Psi(A)) = A$, because Ψ is injective.

Suppose, for a contradiction, that $\Psi(A) \in \mathcal{M}_{\mathrm{Bor}}$. Then $\Psi^{-1}(\Psi(A)) \in \mathcal{M}_{\mathrm{Leb}}$ (Prop 3.3). This gives the required contradiction.

Thus $\Psi(A) \in \mathcal{M}_{\mathrm{Leb}}$ and $\Psi(A) \notin \mathcal{M}_{\mathrm{Bor}}$.

2. One can "count" $\mathcal{M}_{\mathrm{Leb}}$ and $\mathcal{M}_{\mathrm{Bor}}$. Since the Cantor set C is null, any subset of C is null and hence Lebesgue measurable. The Cantor-Lebesgue function maps C onto $[0,1]$ and hence there are "as many" points in C as in $[0,1]$. Thus there are as many subsets of C as there are subsets of $[0,1]$, and hence there are "as many" Lebesgue measurable subsets of $[0,1]$ as there are subsets of $[0,1]$. It can be shown that there "as many" points in $[0,1]$ as there are Borel subsets of $[0,1]$. However, Cantor showed that there are not "as many" points in $[0,1]$ as there are subsets of $[0,1]$. So there are not "as many" Borel sets as there are Lebesgue measurable sets.

Note. In the above, saying that there are "as many" points in A as in B can be interpreted as meaning that there is an injection from B to A, or that there is a surjection from A to B, or that there is a bijection between them—it makes no difference in the cases above. The tricky part of the argument is showing that there "as many" points in $[0,1]$ as there are Borel subsets of $[0,1]$. If you are interested in techniques for such things, you should do B1b Set Theory next year.

Remark. As stated in Prop 3.3,

$$\mathcal{M}_{\mathrm{Leb}} = \{B \setminus N : B \in \mathcal{M}_{\mathrm{Bor}}, N \text{ null}\} = \{A \cup N : A \in \mathcal{M}_{\mathrm{Bor}}, N \text{ null}\}.$$

This is an exercise of moderate difficulty.

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