A3: RINGS AND MODULES, EXAMPLES SHEET 4

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1. Give examples of the following.

(i) A free module over a PID and a linearly independent subset that cannot be extended to a basis.
(ii) A free module over a PID and a minimal generating subset that is not a basis.
(iii) A free module over a PID and a proper submodule of the same rank.

2. Both the conclusions below follow from the Structure Theorem in the lectures notes. The point of this question is to see how the proof simplifies in each case.

(i) Suppose that R is a PID and M is a finitely generated R-module such that $\mathrm{Ann}_R(x) = \{0\}$ for all $x \neq 0_M$. Show that M is free.
(ii) Suppose that G is a finitely generated commutative group. Show that G is isomorphic to a direct sum of cyclic groups.

3. Suppose that R is a commutative unital ring. Show that if every submodule of R (considered as a left R-module) is free then R is a PID.

4. Suppose that $M = \mathbb{C}^3$ is the left $\mathbb{C}[X]$-module endowed with scalar multiplication $\mathbb{C}[X] \to \mathrm{End}(M); p \mapsto (M \to M; v \mapsto p(A)v)$ where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(i) Show that M is cyclic.
(ii) Show that M is isomorphic to the direct sum of three submodules.
(iii) Is M isomorphic to $\mathbb{C}[A]$ as a $\mathbb{C}[X]$-module?
(iv) How do your answers above change if we replace $\mathbb{C}$ by $\mathbb{R}$? with $\mathbb{F}_7$?

5. Let G be the commutative group with generators a, b, and c and relations $2a-16b-8c = 0$, and $4a+24b+8c = 0$. Find $s, r \in \mathbb{N}_0$, natural numbers $d_r \mid \cdots \mid d_1$, and an isomorphism $G \to (\mathbb{Z}/d_r\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/d_1\mathbb{Z}) \oplus \mathbb{Z}^s$.

6.

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(i) Show that the matrix A below has characteristic polynomial $(x-1)^4$ and minimal polynomial $(x-1)^2$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \\ 1 & 1 & -1 & 3 \end{pmatrix}$$

(ii) What are the possible rational canonical forms of a matrix with characteristic polynomial $(x-1)^4$ and minimal polynomial $(x-1)^2$? Which one of these is A?

(iii) Derive the rational canonical form of A by putting the matrix $xI - A$ into Smith normal form.

(iv) What is the Jordan normal form of A?

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