A3: RINGS AND MODULES, EXAMPLES SHEET 2

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1. Suppose that R is an integral domain that is a finite dimensional vector space over a field (contained in R). Show that R is a field.

2. Suppose that $R[X]$ is a PID. Show that R is a field.

3. Show that 2 is irreducible in $\mathbb{Z}[\sqrt{-5}]$, but $\langle 2\rangle$ is not a prime ideal in $\mathbb{Z}[\sqrt{-5}]$.

4. Suppose that R is a non-trivial commutative unital ring in which every proper ideal is prime. Show that R is a field.

5. Suppose that $\mathbb{F}$ is an infinite field. Show $U(\mathbb{F})$ is *not* cyclic.

6. Show that

(i) $X^5 + mX^2 + n$ is irreducible in $\mathbb{Z}[X]$ for $m, n \in \mathbb{Z}$ odd;

(ii) $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ is irreducible in $\mathbb{F}_p[X]$ when p is a prime with $p \equiv 3 \pmod 7$ or $p \equiv 5 \pmod 7$;

(iii) $X^p - X - 1$ is irreducible in $\mathbb{Z}[X]$ for p a prime.

7. Suppose that R is an infinite PID with finitely many units. Show that R has infinitely many maximal ideals.

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